

An application of anisotropic regularization to the existence of weak Pareto minimal points[☆]

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Abstract

Given a function f on R^n , we introduce the concept of anisotropic regularization $f_{\varepsilon, g}$ as a generalization of Tikhonov regularization $f_{\varepsilon}(x) = f(x) + \varepsilon x$. When f is a continuous \mathbf{P}_0 -function on R^n and K is a box in R^n , we study the properties of $f_{\varepsilon, g}$ and the limiting behavior of solutions of a regularized box variational inequality problem $\text{BVI}(f_{\varepsilon, g}, K)$, with emphasis on the existence of weak Pareto minimal points with respect to K . This work generalizes results of Sznajder and Gowda (1998) proved in the setting of nonlinear complementarity problems.

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1. Introduction

Consider a continuous function $f: R^n \rightarrow R^n$ and a rectangular box K in R^n . Then the box variational inequality problem $\text{BVI}(f, K)$ is to find a vector $x^* \in K$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in K. \quad (1.1)$$

When $K = R_+^n$, the above problem reduces to the well-known nonlinear complementarity problem $\text{NCP}(f)$, which is to find a vector $x^* \in R^n$ such that

$$x^* \geq 0, \quad f(x^*) \geq 0, \quad \text{and} \quad \langle f(x^*), x^* \rangle = 0. \quad (1.2)$$

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Throughout this paper we assume f to be a \mathbf{P}_0 - (\mathbf{P})-function, that is, for all $x, y \in R^n$ with $x \neq y$,

$$\max_{\{i: x_i \neq y_i\}} (x_i - y_i)[f_i(x) - f_i(y)] \geq 0 \quad (> 0).$$

Both the NCP and BVI, as well as related problems, have attracted the attention of many researchers and have been extensively studied. Their importance is well documented in the literature, see [3,6,9,10], and references therein.

It is easy to see that $x^* \in R^n$ solves NCP(f) if and only if it is a zero of the nonsmooth function

$$F(x) := x \wedge f(x),$$

where ‘ \wedge ’ is the componentwise minimum. Similarly, $x^* \in R^n$ solves the BVI(f, K) problem if and only if it is a zero of the fixed point map

$$F(x) := x - \Pi_K(x - f(x)),$$

where $\Pi_K(u)$ is the Euclidean projection of an element u onto K . In [8], Gowda and Tawhid provide an extensive list of various perturbations $F(\cdot, \varepsilon)$ of the function F that can be used to generate trajectories, i.e., continuous mappings $\varepsilon \rightarrow x(\varepsilon)$, where $x(\varepsilon)$ is a unique solution to the equation $F(x, \varepsilon) = 0$. Limit points of such a trajectory (if exist) are solutions to the problems (1.1) and (1.2). Let us consider the perturbation

$$F(x, \varepsilon) := x \wedge f_\varepsilon(x),$$

where, for $\varepsilon > 0$, $f_\varepsilon(x) := f(x) + \varepsilon x$ is the Tikhonov regularization of f . We refer to [13,14,18,20] for other types of regularizations. As shown by Subramanian [17], for a monotone f , the entire trajectory converges to the least norm solution of NCP(f) (this result is still valid in the setting of maximal monotone operators via Yosida approximations, see [1, Theorem 3.5.9]). In a general \mathbf{P}_0 case such a result does not hold true, yet, when the solution set SOL(f) of NCP(f) is nonempty and bounded, Ravindran and Gowda [15] showed that the distance between $x(\varepsilon)$ and the solution set SOL(f) tends to zero as $\varepsilon \rightarrow 0$. Earlier, Facchinei and Kanzow [4] proved the same result for a C^1 -function f . In [16] (Theorem 4.1) Sznajder and Gowda show that in case of a \mathbf{P}_0 -function, every accumulation point of $x(\varepsilon)$ is a weak Pareto minimal element of the solution set SOL(f) of NCP(f), and in the linear case, the entire trajectory converges to a weak Pareto minimal element of SOL(f). We point out that the latter result was recently generalized by Facchinei and Pang [5, Theorem 5.10] to a much more general setup of variational inequalities with subanalytic data.

In this paper we concentrate on the framework of box variational inequality problem and introduce the concept of the anisotropic regularization $f_{\varepsilon,g}$ of the function f . This notion itself is a generalization of the Tikhonov regularization. As ε varies over a positive orthant R_{++}^n for $n > 1$, the set $\{x(\varepsilon): \varepsilon > 0\}$, where $x(\varepsilon)$ is a solution of the BVI($f_{\varepsilon,g}, K$), is not a curve anymore. Yet, using the machinery developed in [8,15], we come up with a generalization of Theorem 4.1 in [16] for the anisotropic perturbation in the BVI setup.

2. Preliminaries

In this section we provide basic definitions and facts used in the sequel. Throughout this paper, K denotes a rectangular box, i.e., for the intervals $K_i = [a_i, b_i]$ with $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$, $a_i \leq b_i$,

$$K := K_1 \times \cdots \times K_n.$$

Also, ∂K denotes the topological boundary of K .

Definition 2.1. A continuous function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called *coercive* if for any sequence $\{x^k\} \subset \mathbb{R}^n$ such that $\|x^k\| \rightarrow \infty$, one has $\|f(x^k)\| \rightarrow \infty$. Equivalently, for any compact set $C \subset \mathbb{R}^n$, the inverse image $f^{-1}(C)$ is compact (i.e., f is *proper*).

We will use the following result known in the literature as a subsequence lemma, studied by Facchinei and Kanzow [4], Ravindran and Gowda [15], Tseng [19], and very recently, in the semimonotone setup, by Zhao and Li [21].

Lemma 2.2. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous \mathbf{P}_0 -function and $\{x^k\}$ be a sequence in \mathbb{R}^n such that $\|x^k\| \rightarrow \infty$. Then there exists a subsequence $\{x^{k_j}\}$ and index i such that, either

- (a) $x_i^{k_j} \rightarrow \infty$ and $\{f_i(x^{k_j})\}$ is bounded below, or
- (b) $x_i^{k_j} \rightarrow -\infty$ and $\{f_i(x^{k_j})\}$ is bounded above.

Theorem 2.3 (Banach and Mazur [2]). If $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, locally one-to-one and coercive, then f is a global homeomorphism of \mathbb{R}^n onto itself.

We recall the definition of a (weak) Pareto minimal element.

Definition 2.4. Consider a nonempty set S and an element $x^* \in S$. We say that x^* is a weak Pareto minimal element (Pareto minimal element) of S with respect to a rectangular box K if

$$(x^* - \text{int } K) \cap S = \emptyset$$

(respectively, $(x^* - K) \cap S = \{x^*\}$).

The above concepts are defined with respect to a rectangular box, in particular, the non-negative orthant. In this setting both concepts appear in multi-objective programming [12] and can be defined with respect to any cone.

3. Properties of the anisotropic regularization

In this section we introduce the new concept of anisotropic regularization of a function. This concept involves a perturbation function $g(x, \varepsilon)$. The function g is not arbitrary: we

assume it satisfies the so-called condition (κ): $g : R^n \times R_{++}^n \rightarrow R^n$ is a continuous diagonal mapping, and for any $1 \leq i \leq n$ and $\varepsilon > 0$,

- (a) $g_i(t, \varepsilon)$ is strictly increasing in t ,
- (b) $g_i(0, \varepsilon) = 0$, and for any real t^* , $\lim_{(t, \varepsilon) \rightarrow (t^*, 0)} g_i(t, \varepsilon) = 0$,
- (c) $\lim_{t \rightarrow \pm\infty} g_i(t, \varepsilon) = \pm\infty$.

We say that a mapping $g : R^n \times R_{++}^n \rightarrow R^n$ is *diagonal*, if $g(x, \boldsymbol{\varepsilon}) = (g_1(x_1, \varepsilon_1), \dots, g_n(x_n, \varepsilon_n))$ for any $x = (x_1, x_2, \dots, x_n) \in R^n$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in R_{++}^n$.

Definition 3.1. For a function $f : R^n \rightarrow R^n$, $\boldsymbol{\varepsilon} := (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in R_{++}^n$, and a function g satisfying condition (κ), the *anisotropic regularization* of f is the function

$$f_{\boldsymbol{\varepsilon}, g}(x) := f(x) + g(x, \boldsymbol{\varepsilon}). \quad (3.1)$$

By considering $\boldsymbol{\varepsilon} = \varepsilon \mathbf{e}$, where \mathbf{e} is the vector of ones in R^n , and $g(x, \boldsymbol{\varepsilon}) = \varepsilon x$, we get the classical Tikhonov regularization f_ε .

A modification of Example 2 in [15] gives the function $g(x, \boldsymbol{\varepsilon})$ satisfying condition (κ):

$$g(x, \boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} x + \boldsymbol{\varepsilon}^2 \ln(\mathbf{e} + e^{-|x|/\boldsymbol{\varepsilon}}).$$

All algebraic operations are performed componentwise. The following result summarizes properties of the anisotropic regularization of a \mathbf{P}_0 -function.

Proposition 3.2. For any continuous \mathbf{P}_0 -function $f : R^n \rightarrow R^n$ and any fixed $\boldsymbol{\varepsilon}$ and g satisfying condition (κ), the anisotropic regularization $f_{\boldsymbol{\varepsilon}, g}(\cdot)$ is a continuous coercive \mathbf{P} -function.

Proof. Continuity of $f_{\boldsymbol{\varepsilon}, g}(\cdot)$ is immediate. Assume now that $\{x^k\}$ is such that $\|x^k\| \rightarrow \infty$. By Lemma 2.2, for some index i_* , either

- (a) $x_{i_*}^k \rightarrow \infty$ and $f_{i_*}(x^k) \geq m$, $k \in N$, for some constant m , or
- (b) $x_{i_*}^k \rightarrow -\infty$ and $f_{i_*}(x^k) \leq M$, $k \in N$, for some constant M .

For (a), $(f_{\boldsymbol{\varepsilon}, g}(x^k))_{i_*} \geq m + g_{i_*}(x_{i_*}^k, \varepsilon_{i_*}) \rightarrow \infty$; for (b), $(f_{\boldsymbol{\varepsilon}, g}(x^k))_{i_*} \leq M + g_{i_*}(x_{i_*}^k, \varepsilon_{i_*}) \rightarrow -\infty$. In either case, $\|f_{\boldsymbol{\varepsilon}, g}(x^k)\| \rightarrow \infty$. Hence, $f_{\boldsymbol{\varepsilon}, g}(\cdot)$ is coercive. The \mathbf{P} -property follows from the coordinatewise monotonicity of the function g . \square

Following the ideas presented in the proof of Theorem 10 in [8], Lemma 2.2, and the product property of the Euclidean projector Π_K ,

$$\Pi_{K_1 \times K_2 \times \dots \times K_n}(x) = (\Pi_{K_1}(x_1), \Pi_{K_2}(x_2), \dots, \Pi_{K_n}(x_n)),$$

one can prove the following

Proposition 3.3. For a continuous \mathbf{P}_0 -function f , $\boldsymbol{\varepsilon} \in R_{++}^n$ and a function g satisfying condition (κ), the fixed point mapping corresponding to the anisotropic regularization $f_{\boldsymbol{\varepsilon}, g}$,

$$F_{\boldsymbol{\varepsilon},g}(x) := x - \Pi_K(x - f_{\boldsymbol{\varepsilon},g}(x)) \tag{3.2}$$

is a continuous coercive **P**-function.

Since any **P**-function is one-to-one, the Banach–Mazur theorem implies

Corollary 3.4. *For a continuous \mathbf{P}_0 -function f , $\boldsymbol{\varepsilon} \in R_{++}^n$ and a function g satisfying condition (κ) , the $\text{BVI}(f_{\boldsymbol{\varepsilon},g}, K)$ has a unique solution $x(\boldsymbol{\varepsilon}, g)$.*

We note that the mapping $R_{++}^n \ni \boldsymbol{\varepsilon} \mapsto x(\boldsymbol{\varepsilon}, g)$ is single-valued, but, as we mentioned in Section 1, is not a trajectory anymore. For a fixed function g satisfying condition (κ) , we simply write $x(\boldsymbol{\varepsilon}) := x(\boldsymbol{\varepsilon}, g)$.

There are many methods of creating **P**-functions from \mathbf{P}_0 -functions via regularization procedures. The following example is rooted in [15]. For a function g (satisfying condition (κ)), let μ be a probability Borel measure on R such that $\int_R |t| d\mu(t) < \infty$ and μ does not vanish on any infinite interval. Then, for any continuous function $f : R \rightarrow R$ the following regularization function

$$\hat{F}_{\boldsymbol{\varepsilon},g}(x) := \int_R [x - \Pi_K(x - f_{\boldsymbol{\varepsilon},g}(x) - t\boldsymbol{\varepsilon})] d\mu(t)$$

is a **P**-function. Again, $\boldsymbol{\varepsilon} \in R_{++}^n$, and the integration is performed componentwise. Note that this regularization is more general than that of (3.2).

Another important example (geared toward the nonnegative orthant) is based on the Fischer function $F(x) := x + f(x) - \sqrt{x^2 + f(x)^2}$ [7] and is given by the perturbation

$$F(x, \boldsymbol{\varepsilon}) := x + f(x) - \sqrt{x^2 + f(x)^2 + 2\boldsymbol{\varepsilon}x}$$

Ravindran and Gowda [15] showed that if f is a \mathbf{P}_0 -function, then F is a \mathbf{P}_0 -function and $F(\cdot, \boldsymbol{\varepsilon})$ is a **P**-function. The unique solution $x(\boldsymbol{\varepsilon})$ of $F(x, \boldsymbol{\varepsilon}) = 0$, if exists, satisfies the conditions (see [11])

$$x(\boldsymbol{\varepsilon}) > 0, \quad f(x(\boldsymbol{\varepsilon})) > 0, \quad \text{and} \quad x(\boldsymbol{\varepsilon})_i f_i(x(\boldsymbol{\varepsilon})) = \varepsilon_i > 0 \quad \text{for all } 1 \leq i \leq n.$$

When $\boldsymbol{\varepsilon} = \varepsilon \mathbf{e}$, the trajectory induced by $F(x, \boldsymbol{\varepsilon})$ is known as the central path or interior point trajectory for NCP(f).

4. Weak Pareto minimal property

Here is the main result of this paper.

Theorem 4.1. *Let $K = K_1 \times K_2 \times \dots \times K_n$ be a rectangular box in R^n such that $\text{int } K \neq \emptyset$, and let each K_i be at least one-side bounded. For a given continuous \mathbf{P}_0 -function f , $\boldsymbol{\varepsilon} \in R_{++}^n$ and a function g satisfying condition (κ) , let $x(\boldsymbol{\varepsilon})$ be the unique solution of the regularized problem $\text{BVI}(f_{\boldsymbol{\varepsilon},g}, K)$. Assume $x(\boldsymbol{\varepsilon}^k) \rightarrow x^*$ as $\boldsymbol{\varepsilon}^k \rightarrow 0$ and suppose $\forall k$ $x(\boldsymbol{\varepsilon}^k) \in \text{int } K$. Then x^* is a weak Pareto minimal element of $\text{SOL}(f, K)$ with respect to K .*

Before proceeding with the proof of this theorem, we formulate the following reduction lemma, which shows how to transform the problem to the R_+^n setup.

Lemma 4.2. *The following hold:*

- (a) For $K = K_1 \times K_2 \times \cdots \times K_n$ with $\text{int } K \neq \emptyset$ and one-side bounded, there exists a diagonal signature matrix $D \in R_+^{n \times n}$ (i.e., a matrix whose diagonal entries are equal to ± 1) and a vector $c \in R^n$ such that $D(K) + c \subseteq R_+^n$.
- (b) For any \mathbf{P}_0 - (\mathbf{P})-function f , any diagonal signature matrix D , and vectors p, q in R^n , $x \rightarrow D[f(Dx + p)] + q$ is a \mathbf{P}_0 - (\mathbf{P})-function.
- (c) For $\tilde{K} := D(K) + c$ and $\tilde{f}(x) := D[f(D(x - c))]$, $\text{SOL}(\tilde{f}, \tilde{K}) = D(\text{SOL}(f, K)) + c$.
- (d) For any continuous \mathbf{P}_0 -function f , $\boldsymbol{\varepsilon} \in R_{++}^n$ and g satisfying condition (κ) , the mapping

$$x \rightarrow \tilde{f}_{\boldsymbol{\varepsilon}, g}(x) := D[f_{\boldsymbol{\varepsilon}, g}(D(x - c))] = \tilde{f}(x) + \tilde{g}(x, \boldsymbol{\varepsilon}),$$

with $\tilde{g}(x, \boldsymbol{\varepsilon}) := D[g(D(x - c), \boldsymbol{\varepsilon})]$, is a continuous, coercive \mathbf{P} -function. Moreover, \tilde{g} also satisfies condition (κ) .

Proof. Part (a). Let $K_i := [a_i, b_i]$ for $1 \leq i \leq n$; by our assumption, either a_i or b_i is finite. Define the diagonal entries of D as

$$d_{ii} := \begin{cases} -1 & \text{if } a_i = -\infty, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, $D(K) = \prod_{i=1}^n [\alpha_i, \beta_i]$ such that for $1 \leq i \leq n$, $\alpha_i > -\infty$. Put

$$c_i := \begin{cases} 0 & \text{if } \alpha_i \geq 0, \\ -\alpha_i & \text{if } \alpha_i < 0. \end{cases}$$

Then $D(K) + c \subseteq R_+^n$. Parts (b) and (c) follow directly from the definition of a \mathbf{P}_0 - (\mathbf{P})-function, definition of the BVI, and simple algebraic manipulations. Part (d) follows from (b) and Lemma 2.2. \square

Proof of Theorem 4.1. To simplify notation we use $f^k(x) = f_{\boldsymbol{\varepsilon}^k, g}(x)$ and $x^k := x(\boldsymbol{\varepsilon}^k)$. Lemma 4.2 allows us to consider the rectangular box K in a standard (i.e., R_+^n) position. Assume that x^* is not a weak Pareto minimal element of $\text{SOL}(f, K)$. Then $x^k \rightarrow x^* \in \bar{x} + \text{int } K$ for some $\bar{x} \in \text{SOL}(f, K)$. It follows that for large k , $x_i^k \in \text{int } K_i$ and $x^k - \bar{x} \in \text{int } K$. Also, for every $x \in K$, $\langle f^k(x^k), x - x^k \rangle \geq 0$ and, by specifying coordinates,

$$(f^k(x^k))_i (x_i - x_i^k) \geq 0, \quad \forall x_i \in K_i.$$

Since $x^k \in \text{int } K$, $(f^k(x^k))_i = 0$. Thus, $f(x^k) + g(x^k, \boldsymbol{\varepsilon}^k) = 0$. Observe that $0 \notin \text{int } K$ and $x^* \neq \bar{x}$ imply $x^k \neq \bar{x}$ for large k . By the \mathbf{P} -property of f^k , there exists an index i such that

$$[x^k - \bar{x}]_i [f(x^k) + g(x^k, \boldsymbol{\varepsilon}^k) - (f(\bar{x}) + g(\bar{x}, \boldsymbol{\varepsilon}^k))]_i > 0. \quad (4.1)$$

Letting $k \rightarrow \infty$, by the continuity of f and condition (κ) (b), we get $(x^* - \bar{x})_i (f(x^*) - f(\bar{x}))_i \geq 0$ and $f(x^*) = 0$. Thus, $(x^* - \bar{x})_i (-f(\bar{x}))_i \geq 0$. Taking into account (4.1), we obtain

$$(x^k - \bar{x})_i (-f(\bar{x}) - g(\bar{x}, \boldsymbol{\varepsilon}^k))_i > 0,$$

or

$$-f_i(\bar{x})(x^k - \bar{x})_i - g_i(\bar{x}_i, \varepsilon_i^k)(x^k - \bar{x})_i > 0.$$

Since $\bar{x} \in \text{SOL}(f, K)$, $-g_i(\bar{x}_i, \varepsilon_i^k)(x^k - \bar{x})_i > 0$, and hence, $g_i(\bar{x}_i, \varepsilon_i^k)(x^k - \bar{x})_i < 0$, a contradiction. \square

The nature of weak Pareto minimal points, for both NCP and BVI, is so far not well understood. In particular, stability of these points is an interesting problem.

Let us point out that Zhao and Li [21] obtained a related result (Theorem 5.3) for the nonlinear semimonotone complementarity problem in the context of homotopy continuation trajectory.

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