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Automorphism Invariance of P- and GUS-Properties of Linear Transformations on Euclidean Jordan Algebras

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Generalizing the **P**-property of a matrix, Gowda et al. [Gowda, M. S., R. Sznajder, J. Tao. 2004. Some **P**-properties for linear transformations on Euclidean Jordan algebras. *Linear Algebra Appl.* **393** 203–232] recently introduced and studied **P**- and globally uniquely solvable (**GUS**)-properties for linear transformations defined on Euclidean Jordan algebras. In this paper, we study the invariance of these properties under automorphisms of the algebra and of the symmetric cone. By means of these automorphisms and the concept of a principal subtransformation, we introduce and study ultra and super **P**-(**GUS**)-properties for a linear transformation on a Euclidean Jordan algebra.

Key words: Euclidean Jordan algebra; automorphism; **P**-property; globally uniquely solvable property; complementarity problem; super and ultra **P**-properties
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1. Introduction. A real $n \times n$ matrix M is said to be a **P**-matrix if every principal minor of M is positive. **P**-matrices have found numerous applications in various fields; see, e.g., Berman and Plemmons [3], Cottle et al. [4], and Facchinei and Pang [5]. It is well known that this property can be described in any one of the following ways:

(1) The implication $x \in R^n$, $x * Mx \leq 0 \Rightarrow x = 0$ holds, where $*$ denotes the componentwise product.

(2) For all $q \in R^n$, the linear complementarity problem $\mathrm{LCP}(M, q)$ has a unique solution, that is, there exists a unique $x \in R^n$ such that $x \geq 0$, $Mx + q \geq 0$, and $\langle x, Mx + q \rangle = 0$.

(3) The map $q \mapsto \mathrm{SOL}(M, q)$ is single valued and Lipschitzian on R^n, where $\mathrm{SOL}(M, q)$ denotes the solution set of $\mathrm{LCP}(M, q)$.

While there are numerous other ways of describing the **P**-property of a matrix, we will consider here two automorphism invariance properties that are relevant to our discussion. Consider R^n with the usual inner product $\langle \cdot, \cdot \rangle$ and the componentwise product defined by $(x * y)_i = x_i y_i$ for $i = 1, 2, \ldots, n$, where x_i is the ith component of (column) vector $x \in R^n$. Let $\mathrm{Aut}(R^n)$ denote the set of all invertible matrices A on R^n satisfying the condition $A(x * y) = Ax * Ay$ for all $x, y \in R^n$, and let $\mathrm{Aut}(R^n_+)$ denote the set of all invertible matrices C on R^n satisfying the condition $C(R^n_+) = R^n_+$, where R^n_+ denotes the nonnegative orthant in R^n. We say that elements of $\mathrm{Aut}(R^n)$ are automorphisms of the algebra R^n and those of $\mathrm{Aut}(R^n_+)$ are automorphisms of the cone R^n_+. It is easily seen that $\mathrm{Aut}(R^n)$ consists of permutation matrices, and any element in $\mathrm{Aut}(R^n_+)$ is a product of a permutation matrix and a diagonal matrix with positive diagonal entries. Now we observe that if M is a **P**-matrix, then so are $A^T MA$ and $C^T MC$ for any $A \in \mathrm{Aut}(R^n)$ and $C \in \mathrm{Aut}(R^n_+)$. In other words, the **P**-matrix property is invariant under automorphisms of the algebra and the (nonnegative) cone.

The space R^n together with the usual inner product and the componentwise product is an example of a Euclidean Jordan algebra. A (general) Euclidean Jordan algebra is a finite-dimensional real inner product space V with a bilinear mapping $(x, y) \to x \circ y$ satisfying certain properties (see §2.1). In recent times, Euclidean Jordan algebras have become important in the study of conic optimization; see, e.g., Schmieta and Alizadeh [17]. Two examples of Euclidean Jordan algebras that are heavily studied in the current literature are: (1) the space $\mathcal{S}^n$ of all real symmetric $n \times n$ matrices with inner product $\langle X, Y \rangle := \mathrm{trace}(XY) = \sum_{i,j=1}^n X_{ij} Y_{ij}$ and Jordan product defined by

$$X \circ Y := \tfrac{1}{2}(XY + YX)$$

for $X, Y \in \mathcal{S}^n$, and (2) the space R^n $(n > 1)$ with the usual inner product and the Jordan product defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \bar{y} \end{bmatrix} := \begin{bmatrix} \langle x, y \rangle \\ x_0 \bar{y} + y_0 \bar{x} \end{bmatrix},$$

where $x_0 \in R$ and $\bar{x} \in R^{n-1}$.

In any Euclidean Jordan algebra V, there is the cone of squares $K := \{x \circ x\colon x \in V\}$, which is a self-dual closed convex cone. In such an algebra, one can define the automorphism groups $\mathrm{Aut}(V)$ and $\mathrm{Aut}(K)$ in the following way (Faraut and Korányi [6]): $\Lambda \in \mathrm{Aut}(V)$ if Λ is an algebra automorphism, that is, $\Lambda\colon V \to V$ is an invertible linear transformation satisfying the condition $\Lambda(x \circ y) = \Lambda(x) \circ \Lambda(y)$ for all $x, y \in V$, and $\Gamma \in \mathrm{Aut}(K)$ if Γ is a cone automorphism, that is, $\Gamma\colon V \to V$ is an invertible linear transformation satisfying the condition $\Gamma(K) = K$. In a Euclidean Jordan algebra V, K is a symmetric cone, which means that for any two objects $x, y \in \mathrm{int}(K)$, there is a $\Gamma \in \mathrm{Aut}(K)$ such that $\Gamma(x) = y$.

In Gowda and Song [7], the properties (1)–(3) of a **P**-matrix were extended to a linear transformation defined on $\mathcal{S}^n$; they were further extended to Euclidean Jordan algebras in Gowda et al. [10]. It was shown in Gowda et al. that the generalizations of properties (1)–(3), respectively called the **P**-property, the globally uniquely solvable (**GUS**)-property, and the Lipschitzian **GUS**-property, are all different. A generalization of the positive principal minor property was also introduced in that reference.

One objective of this paper is to study the invariance properties of the above **P**- and **GUS**-properties under the algebra and cone automorphisms. We will show that the properties that are based on the inner product (such as the **GUS**- and Lipschitzian **GUS**-properties) remain invariant under cone automorphisms, but that the **P**-property (which is based on the Jordan product) fails to have this invariance property. However, all the properties that we study remain invariant under algebra automorphisms.

There is another motivation for the present work. It has been shown (see Gowda and Song [7]) that the **GUS**-property implies the **P**-property but that the converse does not hold. As a sufficient condition for the **GUS**-property, Gowda and Song introduced the concept of the $\mathbf{P}_2$-property of a linear transformation on $\mathcal{S}^n$ by means of the following condition:

$$X \geq 0,\ Y \geq 0,\ (X-Y)L(X-Y)(X+Y) \leq 0 \quad \Rightarrow \quad X = Y.$$

Parthasarathy et al. [16] show that a strongly monotone transformation satisfies this property. Gowda et al. [9] show that L has this property if and only if for every invertible $Q \in R^{n\times n}$, every principal subtransformation of $\widehat{L}$, defined by $\widehat{L}(X) := Q^T L(QXQ^T)Q$, has the **P**-(also the **GUS**) property. Because the above $\mathbf{P}_2$-condition is based on the associative property of the ordinary matrix product in $\mathcal{S}^n$, this property cannot be extended to the context of a (general) Euclidean Jordan algebra. However, noting that $\widehat{L} = \Gamma^T L \Gamma$, where Γ (defined by $\Gamma(X) := QXQ^T$) belongs to $\mathrm{Aut}(\mathcal{S}^n_+)$, we extend this property to general Euclidean Jordan algebras via automorphisms of the symmetric cone K. By calling this property the "ultra **P**-property," we show that the ultra **P**-property implies the **GUS**-property. In addition, we show that the Lipschitzian **GUS**-property implies the ultra **P**-property under certain conditions. We also define a related "super **P**-property" by using algebra automorphisms and study some of its properties.

2. Preliminaries.

2.1. Euclidean Jordan algebras.

In this subsection, we briefly recall some concepts, properties, and results from Euclidean Jordan algebras. Most of these can be found in Faraut and Korányi [6], Schmieta and Alizadeh [17], and Gowda et al. [10].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle \cdot,\cdot \rangle)$, where $(V, \langle \cdot,\cdot \rangle)$ is a finite-dimensional inner product space over R and $(x, y) \mapsto x \circ y\colon V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in V$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$, where $x^2 := x \circ x$, and
(iii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$ for all $x, y, z \in V$.

In addition, we assume that there is an element $e \in V$ (called the *unit element*) such that $x \circ e = x$ for all $x \in V$.

Henceforth, *we assume that V is a Euclidean Jordan algebra* and call $x \circ y$ the Jordan product of x and y. In V, the set of squares

$$K := \{x \circ x\colon x \in V\}$$

is a *symmetric cone* (see, Faraut and Korányi [6, p. 46]). This means that K is a self-dual closed convex cone and for any two elements $x, y \in \mathrm{int}(K)$, there exists an invertible linear transformation $\Gamma\colon V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$.

For an element $z \in V$, we write

$$z \geq 0 \quad \text{if and only if} \quad z \in K,$$

and $z \leq 0$ when $-z \geq 0$. We write $z > 0$ if $z \in \mathrm{int}(K)$.

For $x \in V$, we define $m(x) := \min\{k > 0\colon \{e, x, \ldots, x^k\}$ is linearly dependent$\}$ and *rank* of V by $r = \max\{m(x)\colon x \in V\}$. An element $c \in V$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, e_2, \ldots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \quad \text{if } i \neq j, \qquad \text{and} \qquad \sum_1^m e_i = e.$$

Note that $\langle e_i, e_j\rangle = \langle e_i \circ e_j, e\rangle = 0$ whenever $i \neq j$.

Theorem 2.1 (The Spectral Decomposition Theorem; Faraut and Korányi [6]). *Let V be a Euclidean Jordan algebra with rank r. Then, for every $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{1}$$

The numbers λ_i are called the eigenvalues of x, and the expression $\lambda_1 e_1 + \cdots + \lambda_r e_r$ is the spectral decomposition (or the spectral expansion) of x.

Example 2.1. Let $\mathcal{S}^n$ be the set of all $n \times n$ real symmetric matrices with the inner and Jordan product given by

$$\langle X, Y\rangle := \operatorname{trace}(XY) \qquad \text{and} \qquad X \circ Y := \frac{1}{2}(XY + YX).$$

In this setting, the cone of squares $\mathcal{S}^n_+$ is the set of all positive semidefinite matrices in $\mathcal{S}^n$.

Example 2.2. Consider R^n $(n > 1)$, where any element x is written as

$$x = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}$$

with $x_0 \in R$ and $\bar{x} \in R^{n-1}$. The inner product in R^n is the usual inner product. The Jordan product $x \circ y$ in R^n is defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \bar{y} \end{bmatrix} := \begin{bmatrix} \langle x, y\rangle \\ x_0\bar{y} + y_0\bar{x} \end{bmatrix}.$$

We shall denote this Euclidean Jordan algebra $(R^n, \circ, \langle\cdot,\cdot\rangle)$ by $\mathcal{L}^n$. In this algebra, the cone of squares, denoted by $\mathcal{L}^n_+$, is called the *Lorentz cone* (or the second-order cone). It is given by

$$\mathcal{L}^n_+ = \{x\colon \|\bar{x}\| \leq x_0\}.$$

In a Euclidean Jordan algebra V for a given $x \in V$, we define the corresponding *Lyapunov transformation* $L_x\colon V \to V$ by

$$L_x(z) = x \circ z.$$

We say that elements x and y *operator commute* if L_x and L_y commute, i.e.,

$$L_x L_y = L_y L_x.$$

It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame (see, Faraut and Korányi [6, Lemma X.2.2] or Schmieta and Alizadeh [17, Theorem 27]).

We recall the following from Gowda et al. [10]:

Proposition 2.1. *For $x, y \in V$, the following conditions are equivalent*:
(1) $x \geq 0$, $y \geq 0$, *and* $\langle x, y\rangle = 0$, *and*
(2) $x \geq 0$, $y \geq 0$, *and* $x \circ y = 0$.
In each case, elements x and y operator commute.

The Peirce decomposition. Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V\colon x \circ e_i = x\} = Re_i,$$

and when $i \neq j$,

$$V_{ij} := \left\{x \in V\colon x \circ e_i = \tfrac{1}{2}x = x \circ e_j\right\}.$$

Then, we have the following:

THEOREM 2.2 (FARAUT AND KORÁNYI [6, THEOREM IV.2.1]). *The space V is the orthogonal direct sum of spaces V_{ij} $(i \le j)$. Furthermore,*

$$\begin{aligned} &V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj}, \\ &V_{ij} \circ V_{jk} \subset V_{ik} \quad \text{if } i \neq k, \\ &V_{ij} \circ V_{kl} = \{0\} \quad \text{if } \{i,j\} \cap \{k,l\} = \varnothing. \end{aligned}$$

Thus, given any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij},$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$.

Simple Jordan algebras and the structure theorem. A Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two Euclidean Jordan algebras. The classification theorem (Faraut and Korányi [6, Chapter V]) says that every simple Euclidean Jordan algebra is isomorphic to one of the following:

(1) The algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices (Example 2.1),

(2) The algebra $\mathcal{L}^n$ (Example 2.2),

(3) The algebra $\mathcal{H}_n$ of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = (1/2) \cdot (XY + YX)$,

(4) The algebra $\mathcal{Q}_n$ of all $n \times n$ quaternion Hermitian matrices with trace inner product and $X \circ Y = (1/2)(XY + YX)$,

(5) The algebra $\mathcal{O}_3$ of all 3×3 octonion Hermitian matrices with trace inner product and $X \circ Y = (1/2) \cdot (XY + YX)$.

The following result characterizes all Euclidean Jordan algebras.

THEOREM 2.3 (FARAUT AND KORÁNYI [6, PROPOSITIONS III.4.4 AND III.4.5, THEOREM V.3.7]). *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

2.2. Principal subtransformations and principal minors. Given a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V, we define

$$V^{(l)} = V(e_1 + e_2 + \cdots + e_l, 1) := \{x \in V : x \circ (e_1 + e_2 + \cdots + e_l) = x\} \tag{2}$$

for $1 \le l \le r$. It is known that $V^{(l)}$ (called the eigenspace of $e_1 + e_2 + \cdots + e_l$) is a subalgebra of V with rank l (Faraut and Korányi [6, Proposition IV.1.1]). By means of Peirce decomposition, we have the following representation (see e.g., Gowda et al. [10]):

$$V^{(l)} = Re_1 + Re_2 + \cdots + Re_l + \sum_{i<j\le l} V_{ij}. \tag{3}$$

Let $P^{(l)}$ denote the orthogonal projection from V onto $V^{(l)}$. For a linear transformation $L: V \to V$, let

$$L_{\{e_1, e_2, \ldots, e_l\}} := P^{(l)} L : V^{(l)} \to V^{(l)}.$$

We call $L_{\{e_1, e_2, \ldots, e_l\}}$ a *principal subtransformation* of L. The determinant of this transformation is called a *principal minor* of L.

2.3. Various P- and GUS-properties. In Gowda et al. [10], generalizing conditions (1)–(3) of the introduction, various **P**- and **GUS**-properties were introduced. We recall these definitions.

Consider a linear transformation $L: V \to V$. We say that L has

(1) The Jordan **P**-property if $x \circ L(x) \leq 0 \Rightarrow x = 0$;

(2) The **P**-property if

$$\left.\begin{array}{r} x \text{ and } L(x) \text{ operator commute} \\ x \circ L(x) \leq 0 \end{array}\right\} \Rightarrow x = 0.$$

Given L and $q \in V$, we define the linear complementarity problem LCP(L, q) as follows: Find $x \in V$ such that

$$x \in K, \qquad L(x) + q \in K, \qquad \text{and} \qquad \langle x, L(x) + q \rangle = 0.$$

In the above condition, in view of Proposition 2.1, we can replace $\langle x, L(x) + q\rangle = 0$ by $x \circ (L(x) + q) = 0$. We denote the set of all solutions of LCP(L, q) by SOL(L, q).

We say that L has

(a) The $\mathbf{R}_0$-property if SOL$(L, 0) = \{0\}$;

(b) The **Q**-property if SOL$(L, q) \neq \varnothing$ for all $q \in V$;

(c) The **GUS**-property if for all $q \in V$, LCP(L, q) has a unique solution;

(d) The Lipschitzian **GUS**-property if L has the **GUS**-property and the solution map $q \to$ SOL(L, q) is Lipschitzian on V; and

(e) The positive principal minor (positive **PM**)-property if every principal minor of L is positive.

Finally, we say that L is monotone (strongly monotone) if $\langle L(x), x\rangle \geq 0$ (respectively, >0) for all $0 \neq x \in V$. The following implications are known (see Gowda et al. [10] and Figure 1):

$$\text{strongly monotone} \;\Rightarrow\; \text{Lipschitzian } \mathbf{GUS} \;\Rightarrow\; \mathbf{GUS} \;\Rightarrow\; \mathbf{P} \;\Rightarrow\; \mathbf{R}_0,$$

$$\text{Jordan } \mathbf{P} \;\Rightarrow\; \mathbf{P} \;\Rightarrow\; \mathbf{Q}, \qquad \text{and}$$

$$\text{Lipschitzian } \mathbf{GUS} \;\Rightarrow\; \mathbf{GUS} + \text{positive } \mathbf{PM}.$$

3. The facial structure of symmetric cones. Let K be the cone of squares in a Euclidean Jordan algebra V. Recall that a nonempty convex set $F \subseteq K$ is a face of K if

$$x, y \in K, \; \tfrac{1}{2}(x + y) \in F \quad \Rightarrow \quad x, y \in F.$$

It is easy to see that any face of K is necessarily a closed convex (sub)cone in K. Obviously, $F = \{0\}$ and $F = K$ are faces of K. In this section, we describe the faces of K (Theorem 3.1 below). We believe that the results of this section are known. For lack of precise references and to be complete, we have provided the proofs; the recent paper of Malik [14] also contains similar results. In what follows, for an element $a \in K$, we let $\Phi(a)$ denote the smallest face of K containing a. We have the following result of Barker and Schneider [2] proved in the setting of general closed convex cones.

Proposition 3.1. *Let F be a face of K and $a \in \operatorname{ri} F$. Then, $F = \Phi(a)$.*

Proof. Certainly, $F \supseteq \Phi(a)$. Let $x \in F$ be arbitrary. Because $a \in \operatorname{ri} F$, there exists $z \in F$ such that a is a convex combination of x and z. Because $a \in \Phi(a)$ and $\Phi(a)$ is a face of K, we have $x \in \Phi(a)$. Thus, $F \subseteq \Phi(a)$ and hence $F = \Phi(a)$. □

Proposition 3.2. *Suppose that $x \geq 0$ and let $x = \sum_1^r x_i e_i + \sum_{i<j} x_{ij}$ be its Peirce decomposition. If $x_k = 0$ for some index k, then $\sum_{k<j} x_{kj} + \sum_{i<k} x_{ik} = 0$. In particular, if $x \in K \cap (\sum_{i<j} V_{ij})$, then $x = 0$.*

Proof. Suppose that $x_k = 0$. Then, $\langle x, e_k\rangle = x_k \|e_k\|^2 = 0$. As $x, e_k \geq 0$, we must have $x \circ e_k = 0$ by Proposition 2.1. This leads, via Theorem 2.2, to $0 = (1/2)(\sum_{k<j} x_{kj} + \sum_{i<k} x_{ik})$. □

Proposition 3.3. *If $x \geq 0$, $y \geq 0$, and $x + y \in V^{(l)}$, then $x, y \in V^{(l)}$.*

Proof. By Peirce decomposition,

$$x = \sum_{j=1}^{r} \mu_j e_j + \sum_{i<j} x_{ij} \qquad \text{and} \qquad y = \sum_{j=1}^{r} \nu_j e_j + \sum_{i<j} y_{ij}.$$

Hence,

$$x+y=\sum_{j=1}^{l}(\mu_j+\nu_j)e_j+\sum_{j=l+1}^{r}(\mu_j+\nu_j)e_j+\sum_{i<j}(x_{ij}+y_{ij}).$$

Because $x+y\in V^{(l)}$, by (3), we have $\mu_k+\nu_k=0$ for $l+1\le k\le r$. Because $\mu_k\|e_k\|^2=\langle x, e_k\rangle$, μ_k and (similarly) ν_k are nonnegative. It follows that $\mu_k=\nu_k=0$ whenever $l+1\le k\le r$. Thus, $x=\sum_{i=1}^{l}\mu_i e_i+\sum_{i<j}x_{ij}$. In view of the previous proposition, for any index k with $l+1\le k\le r$, we have $\sum_{k<j}x_{kj}+\sum_{i<k}x_{ik}=0$. Hence, $x=\sum_{i=1}^{l}\mu_i e_i+\sum_{i<j\le l}x_{ij}$ and so $x\in V^{(l)}$ (similarly for y). This proves the result. □

Theorem 3.1. *Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V and an index $1\le l\le r$. Then, $K^{(l)}:=$ cone of squares in $V^{(l)}$ is a face of K. Conversely, let $F\neq\{0\}$ be a face of K. Then, there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V such that $F=K^{(l)}$ for some $1\le l\le r$.*

Proof. Fix an index l such that $1\le l\le r$ and consider $K^{(l)}$. Clearly, $K^{(l)}\subseteq V^{(l)}\cap K$. To see the reverse inclusion, pick any $x\in V^{(l)}\cap K$. Then, for any $z\in K^{(l)}$, we have $\langle z, x\rangle\ge 0$ (as $x, z\in K$). Because $K^{(l)}$ is a self-dual cone in $V^{(l)}$, $x\in K^{(l)}$. Hence, $K^{(l)}=V^{(l)}\cap K$.

We now show that $K^{(l)}$ is a face. Let $x, y\in K$ and $(1/2)(x+y)\in K^{(l)}$. Then, $x+y\in K^{(l)}\subseteq V^{(l)}$ and by Proposition 3.3, x and y are in $V^{(l)}$. As x and y are also in K, we have $x, y\in K^{(l)}$.

Now for the converse. As $F\neq\{0\}$, we can pick x such that $0\neq x\in \operatorname{ri}F$. By Proposition 3.1, $F=\Phi(x)$. Corresponding to x, there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V such that

$$x=\sum_{1}^{r}\lambda_i e_i=\lambda_1 e_1+\cdots+\lambda_l e_l+0e_{l+1}+\cdots+0e_r,$$

where $\lambda_1, \ldots, \lambda_l>0$. It is immediate that $x\in K^{(l)}=V^{(l)}\cap K$. We show now that $x\in\operatorname{ri}K^{(l)}$. To see this, let $0\neq y\in K^{(l)}$. Because $y\in V^{(l)}$, by Equation (3), $y=\sum_1^l\mu_i e_i+\sum_{i<j\le l}y_{ij}$. By Proposition 3.2, for some index i, μ_i is nonzero. For this index i, $0\le\langle y, e_i\rangle=\mu_i\|e_i\|^2$. Then, $\mu_i>0$ and so $\langle x, y\rangle=\sum_1^l\lambda_i\mu_i>0$. Because y is arbitrary, we must have $x\in\operatorname{ri}K^{(l)}$. By the first part of the proof, $K^{(l)}$ is already a face of K, and so by Proposition 3.1, $\Phi(x)=K^{(l)}$. Because $F=\Phi(x)$, we get $F=K^{(l)}$. □

4. Algebra and cone automorphisms. This section deals with (Euclidean Jordan) algebra and cone automorphisms. The results of this section will be used in the subsequent sections dealing with the main results. They are also of independent interest.

A linear transformation Λ: $V\to V$ is said to be an *algebra automorphism* if Λ is invertible and $\Lambda(x\circ y)=\Lambda(x)\circ\Lambda(y)$ for all $x, y\in V$. The set of all automorphisms of V is denoted by $\operatorname{Aut}(V)$.

A linear transformation Γ: $V\to V$ is said to be a (*symmetric*) *cone automorphism* if $\Gamma(K)=K$. Because $\operatorname{int}K\neq\varnothing$, such a transformation is necessarily invertible. We denote the set of all automorphisms of K by $\operatorname{Aut}(K)$. It is immediate that $\operatorname{Aut}(V)\subseteq\operatorname{Aut}(K)$.

Recall that a linear transformation is orthogonal on V if it preserves the inner product in V. Let $\operatorname{Orth}(V)$ denote the set of all orthogonal transformations on V. It should be noted that $\operatorname{Aut}(V)$ need not be contained in $\operatorname{Orth}(V)$; for an example, see, Faraut and Korányi [6, p. 56]. On the other hand, if the inner product is a constant multiple of the trace of the Jordan product (that is, there is a positive constant c such that $\langle x, y\rangle=c\operatorname{tr}(x\circ y)$ for all $x, y\in V$), then $\operatorname{Aut}(V)\subseteq\operatorname{Orth}(V)$. In particular, this inclusion holds if V is simple (Faraut and Korányi [6, Proposition III.4.1]).

Example 4.1. Consider $V=\mathcal{S}^n$. In this case, it is known (see Schneider [18]) that corresponding to any $\Gamma\in\operatorname{Aut}(\mathcal{S}^n_+)$, there exists an invertible matrix $Q\in R^{n\times n}$ such that

$$\Gamma(X)=QXQ^T\quad(X\in\mathcal{S}^n).$$

Also, for $\Lambda\in\operatorname{Aut}(\mathcal{S}^n)$, there exists an orthogonal matrix U such that

$$\Lambda(X)=UXU^T\quad(X\in\mathcal{S}^n).$$

Example 4.2. Consider $V=\mathcal{L}^n$. In this case, it is known (see Loewy and Schneider [13]) that if $n\times n$ matrix Γ belongs to $\operatorname{Aut}(\mathcal{L}^n_+)$, then there exists $\mu>0$ such that

$$\Gamma^T J_n\Gamma=\mu J_n,$$

where $J_n = \text{diag}(1, -1, -1, \ldots, -1)$. In particular, if $\Lambda \in \text{Aut}(\mathcal{L}^n)$, then (because $\Lambda(e) = e$), it can be easily seen that

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix},$$

where D: $R^{n-1} \to R^{n-1}$ is an orthogonal matrix.

We now state two elementary results.

Proposition 4.1 (Koecher [12, p. 10]). *Let $\Gamma \in \text{Aut}(K)$. Then, Γ^{-1} and $\Gamma^T \in \text{Aut}(K)$.*

Proof. We start by noting that Γ^{-1} and Γ^T are linear. Because $\Gamma(K) = K$, we have $K = \Gamma^{-1}(K)$ and so $\Gamma^{-1} \in \text{Aut}(K)$. Now let $x \in K$ be arbitrary. Then, for any $y \in K$, $\langle \Gamma^T(x), y\rangle = \langle x, \Gamma(y)\rangle \geq 0$ (because $\Gamma(y) \in K$). Thus, $\Gamma^T(x) \in K$ (as K is self-dual). We conclude that $\Gamma^T(K) \subseteq K$ for any $\Gamma \in \text{Aut}(K)$. Thus,

$$K \subseteq (\Gamma^T)^{-1}(K) = (\Gamma^{-1})^T(K) \subseteq K,$$

as $\Gamma^{-1} \in \text{Aut}(K)$. It follows that $K = (\Gamma^T)^{-1}(K)$; hence $(\Gamma^T)^{-1} \in \text{Aut}(K)$, which implies that $\Gamma^T \in \text{Aut}(K)$. □

Remark 4.1. If $\Lambda \in \text{Aut}(V)$, then obviously $\Lambda^{-1} \in \text{Aut}(V)$. However, Λ^T need not be in $\text{Aut}(V)$, as the following example shows.

Example 4.3. Let $V = R^2$ with componentwise product. Let the inner product be defined by

$$\langle x, y\rangle = x_1y_1 + 2x_2y_2,$$

where $x = [x_1, x_2]^T$ and $y = [y_1, y_2]^T$. It is easily seen that

$$\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \text{Aut}(V) \qquad \text{and} \qquad \Lambda^T = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix} \notin \text{Aut}(V).$$

We also note that $(\Lambda^T)^{-1}(e_1) = \Lambda(e_1)/2$ and $(\Lambda^T)^{-1}(e_2) = 2\Lambda(e_2)$, where $e_1 = [1, 0]^T$ and $e_2 = [0, 1]^T$ are (the only) primitive idempotents in V.

Proposition 4.2. *Let $\Lambda \in \text{Aut}(V)$ and $\{e_1, e_2, \ldots, e_r\}$ be any Jordan frame in V. Then,*
(1) *$\{\Lambda(e_1), \Lambda(e_2), \ldots, \Lambda(e_r)\}$ is also a Jordan frame in V;*
(2) *For each i, there exists a positive number θ_i such that $(\Lambda^T)^{-1}(e_i) = \theta_i\Lambda(e_i)$.*

Proof. Because Λ preserves the Jordan product in V, item (1) follows from the definition of a Jordan frame. We prove (2). Consider indexes i and j. Because $\Lambda \in \text{Aut}(V) \subseteq \text{Aut}(K)$, by the previous proposition, $\Lambda(e_i) \geq 0$ and $(\Lambda^T)^{-1}(e_j) \geq 0$. Let $u := \Lambda(e_i) \circ (\Lambda^T)^{-1}(e_j)$. Then, for any $x \in V$,

$$\begin{aligned}\langle u, \Lambda(x)\rangle &= \langle \Lambda(e_i) \circ (\Lambda^T)^{-1}(e_j), \Lambda(x)\rangle \\ &= \langle (\Lambda^T)^{-1}(e_j), \Lambda(e_i) \circ \Lambda(x)\rangle \\ &= \langle (\Lambda^T)^{-1}(e_j), \Lambda(e_i \circ x)\rangle \\ &= \langle e_j, e_i \circ x\rangle \\ &= \langle e_j \circ e_i, x\rangle.\end{aligned}$$

It follows that $\langle \Lambda^T(u), x\rangle = \langle e_i \circ e_j, x\rangle$ for all $x \in V$ and hence $\Lambda^T(u) = e_i \circ e_j = \delta_{ij}e_j$. Thus, $u = \delta_{ij}(\Lambda^T)^{-1}(e_j)$. This proves the equality

$$\Lambda(e_i) \circ (\Lambda^T)^{-1}(e_j) = \delta_{ij}(\Lambda^T)^{-1}(e_j) \quad \forall\, i, j. \tag{4}$$

We now claim that for all i, $\Lambda(e_i)$ and $(\Lambda^T)^{-1}(e_i)$ operator commute. To see this, fix i, put $x = \Lambda(e_i)$, $y = (\Lambda^T)^{-1}(e_i)$, and consider the identity (see Faraut and Korányi [6, Proposition II.1.1])

$$[L_x, L_{y^2}] + 2[L_y, L_{x\circ y}] = 0,$$

where $[A, B] := AB - BA$. Because $x \circ y = y$, the second term in the above expression is zero. Thus, the first term is also zero proving the operator commutativity of x and y^2. This means that there is a common Jordan frame $\{f_1, f_2, \ldots, f_r\}$ with respect to which x and y^2 have their spectral representations: $x = \sum \mu_i f_i$ and $y^2 = \sum \nu_i f_i$. Because $y \geq 0$, we may write $y = \sqrt{y^2} = \sum \sqrt{\nu_i} f_i$. This means that x and y have their spectral representations with respect to a common Jordan frame, and so x and y operator commute. To show that y is a positive scalar multiple of x, let for simplicity, $i=1$. Let $g_1=\Lambda(e_1)$. Because $(\Lambda^T)^{-1}(e_1)$ operator commutes with g_1, there is a common Jordan frame $\{u_1,u_2,\ldots,u_r\}$ with respect to which $(\Lambda^T)^{-1}(e_1)$ and g_1 will have their spectral representations: $g_1=\sum\alpha_i u_i$ and $(\Lambda^T)^{-1}(e_1)=\sum\beta_i u_i$. Because $\Lambda\in\mathrm{Aut}(V)$, g_1 is a primitive idempotent; hence, α_i is zero for all indexes i except one, (say) $i=1$. Then, $g_1=u_1$. It follows from $(\Lambda^T)^{-1}(e_1)=\sum\beta_i u_i$ that $g_1\circ(\Lambda^T)^{-1}(e_1)=\beta_1 u_1$. Because $u_1=g_1=\Lambda(e_1)$, we have $\Lambda(e_1)\circ(\Lambda^T)^{-1}(e_1)=\beta_1\Lambda(e_1)$. From Equation (4), we have $(\Lambda^T)^{-1}(e_1)=\beta_1\Lambda(e_1)$. Note that $\beta_1\geq 0$ as $(\Lambda^T)^{-1}(e_1)\geq 0$. Also, $\beta_1\neq 0$ as $(\Lambda^T)^{-1}(e_1)=0$ would imply that $e_1=0$, which is a contradiction. Hence, we have item (2). □

In the next proposition, we describe the effect of an algebra automorphism on the eigenspace $V^{(l)}$ and the principal subtransformations of a linear transformation.

Proposition 4.3. *Suppose that $\{e_1,e_2,\ldots,e_r\}$ is a Jordan frame and $\Lambda\in\mathrm{Aut}(V)$. Let $\Lambda(e_i)=f_i$ for $1\leq i\leq r$. Let $V^{(l)}$ be the eigenspace of $e_1+e_2+\cdots+e_l$ (see §2) and $W^{(l)}$ be the eigenspace of $f_1+f_2+\cdots+f_l$. Let $P^{(l)}$ ($Q^{(l)}$) denote the orthogonal projection onto $V^{(l)}$ (respectively, $W^{(l)}$). For a given L: $V\to V$, let $\widetilde{L}=\Lambda^T L\Lambda$. Then, the following statements hold*:

(i) $W^{(l)}=\Lambda(V^{(l)})$;

(ii) $\Lambda^T(W^{(l)})=V^{(l)}$ *and* $\Lambda^T((W^{(l)})^\perp)=(V^{(l)})^\perp$;

(iii) $\Lambda^T Q^{(l)}=P^{(l)}\Lambda^T$; *and*

(iv) $(\Lambda^T L\Lambda)_{\{e_1,e_2,\ldots,e_l\}}=\Lambda^T L_{\{f_1,f_2,\ldots,f_l\}}\Lambda$ *on* $V^{(l)}$.

Proof. (i)

$$\begin{aligned}\Lambda(V^{(l)}) &= \Lambda\{y\in V\colon y\circ(e_1+e_2+\cdots+e_l)=y\}\\ &= \Lambda\{\Lambda^{-1}x\in V\colon (\Lambda^{-1}x)\circ(e_1+e_2+\cdots+e_l)=\Lambda^{-1}x\}\\ &= \{x\in V\colon (\Lambda^{-1}x)\circ(e_1+e_2+\cdots+e_l)=\Lambda^{-1}x\}\\ &= \{x\in V\colon x\circ(f_1+f_2+\cdots+f_l)=x\}\quad(\text{because } \Lambda\in\mathrm{Aut}(V))\\ &= W^{(l)}.\end{aligned}$$

(ii) Let $u\in V$. Then, for any $x\in V$, we have

$$\begin{aligned}\langle\Lambda^T u\circ(e_1+e_2+\cdots+e_l),x\rangle &= \langle\Lambda^T u,(e_1+e_2+\cdots+e_l)\circ x\rangle\\ &= \langle u,\Lambda(e_1+e_2+\cdots+e_l)\circ\Lambda(x)\rangle\\ &= \langle u,(f_1+f_2+\cdots+f_l)\circ\Lambda(x)\rangle\\ &= \langle u\circ(f_1+f_2+\cdots+f_l),\Lambda(x)\rangle\\ &= \langle\Lambda^T(u\circ(f_1+f_2+\cdots+f_l)),x\rangle.\end{aligned}$$

Hence, $\Lambda^T(u)\circ(e_1+e_2+\cdots+e_l)=\Lambda^T(u\circ(f_1+f_2+\cdots+f_l))$. From this, it follows that $u\in W^{(l)}$ if and only if $\Lambda^T u\in V^{(l)}$. This proves the first part of (ii). For the second part, observe that $\langle\Lambda^T x,y\rangle=\langle x,\Lambda(y)\rangle$ for any $x,y\in V$. If $x\in(W^{(l)})^\perp$, then for any $y\in V^{(l)}$, we have $\langle\Lambda^T x,y\rangle=0$ because of (i). Hence, $\Lambda^T x\in(V^{(l)})^\perp$. A similar argument shows that if $\Lambda^T x\in(V^{(l)})^\perp$ for some x, then $x\in(W^{(l)})^\perp$. Thus, we have (ii).

(iii) Let $x\in V$ be written as $x=u+v$, where $u\in W^{(l)}$ and $v\in(W^{(l)})^\perp$. Then, $\Lambda^T x=\Lambda^T u+\Lambda^T v$. By (ii), $\Lambda^T u\in V^{(l)}$ and $\Lambda^T v\in(V^{(l)})^\perp$. Hence, $P^{(l)}\Lambda^T x=\Lambda^T u$. This implies that $(\Lambda^T)^{-1}P^{(l)}\Lambda^T x=u$. However, $Q^{(l)}x=u$. Hence, $(\Lambda^T)^{-1}P^{(l)}\Lambda^T=Q^{(l)}$ proving (iii).

(iv) Let $z\in V^{(l)}$. Then,

$$\begin{aligned}\Lambda^T L_{\{f_1,f_2,\ldots,f_l\}}\Lambda(z) &= \Lambda^T Q^{(l)}(L(\Lambda(z)))\\ &= P^{(l)}\Lambda^T(L(\Lambda(z)))=P^{(l)}(\widetilde{L})(z)=(\widetilde{L})_{\{e_1,e_2,\ldots,e_l\}}(z).\end{aligned}$$

This proves (iv). □

Remark 4.2. (a) In addition to the above, we note that $u\geq 0$ in $V^{(l)}$ if and only if $\Lambda(u)\geq 0$ in $W^{(l)}$, and $w\geq 0$ in $W^{(l)}$ if and only if $\Lambda^T w\geq 0$ in $V^{(l)}$.

(b) Also, $x\geq 0$ in V implies $P^{(l)}(x)\geq 0$ in $V^{(l)}$. In fact, because $V^{(l)}$ is an algebra, $K^{(l)}=\{y\circ y\colon y\in V^{(l)}\}\subseteq V^{(l)}$, so

$$\langle P^{(l)}(x),y\circ y\rangle=\langle x,P^{(l)}(y\circ y)\rangle=\langle x,y\circ y\rangle\geq 0\quad\text{for all } y\in V^{(l)}.$$

Thus, $P^{(l)}(x)\geq 0$ in $V^{(l)}$.

If $\{e_1, e_2, \ldots, e_r\}$ and $\{f_1, f_2, \ldots, f_r\}$ are any two Jordan frames in a simple algebra V, then it is known that there is an algebra automorphism $\Lambda \in \mathrm{Aut}(V)$ such that $\Gamma(e_i) = f_i$ for all i; see Faraut and Korányi [6, Theorem IV.2.5]. As the following example shows, this result fails if V is not simple.

Example 4.4. Consider the product algebra $V = R \times \mathcal{S}^2$ with the Jordan frame $\{e_1, e_2, e_3\}$, where $e_1 = (1, 0)$, $e_2 = (0, E_1)$, and $e_3 = (0, E_2)$; E_1 (E_2) being the matrix in $\mathcal{S}^2$ with one in the $(1, 1)$ (respectively, $(2, 2)$) slot and zeros elsewhere. We claim that there is no automorphism $\Lambda \in \mathrm{Aut}(V)$ such that $\Lambda(e_1) = e_2$. Assuming the existence of such a Λ, we may write for all $(\lambda, X) \in V$,

$$\Pi_1 \Lambda(\lambda, X) = \alpha\lambda + \langle B, X\rangle,$$

where $\alpha \in R$ and $B \in \mathcal{S}^2$ and Π_1 denotes the projection mapping which takes $(t, X) \in V$ to t. As $\Lambda(e_1) = e_2$, α must be zero. Then, the mapping $X \to (0, X) \to \Pi_1\Lambda(0, X) = \langle B, X\rangle$ satisfies the condition

$$\langle B, X \circ Y\rangle = \langle B, X\rangle\langle B, Y\rangle$$

for all $X, Y \in \mathcal{S}^2$. Because B is orthogonally similar to a diagonal matrix D, the above equality holds for D in place of B. As $\Lambda(1, I) = (1, I)$, we must have $\Pi_1\Lambda(1, I) = 1$, and so $\langle B, I\rangle = 1$ which further implies that $\langle D, I\rangle = 1$. On the other hand, if

$$E_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then $\langle D, I\rangle = \langle D, E_{12} \circ E_{12}\rangle = \langle D, E_{12}\rangle\langle D, E_{12}\rangle = 0$ yields a contradiction.

The following result shows that "up to a permutation" one Jordan frame can be mapped to another in any algebra.

Proposition 4.4. *Let $\{e_1, e_2, \ldots, e_r\}$ and $\{f_1, f_2, \ldots f_r\}$ be any two Jordan frames in V. Then, there exists a $\Lambda \in \mathrm{Aut}(V)$ such that $\{\Lambda(e_1), \Lambda(e_2), \ldots, \Lambda(e_r)\} = \{f_1, f_2, f \ldots, f_r\}$, that is, there exists a permutation σ of $\{1, 2, \ldots, r\}$ such that $\Lambda(e_i) = f_{\sigma(i)}$ for all i.*

Proof. Using Theorem 2.3, we may write $V = V_1 \times V_2 \times \cdots \times V_k$, where each V_i is simple. For notational simplicity, we let $k = 2$ and $r_1 = \mathrm{rank}(V_1)$, $r_2 = \mathrm{rank}(V_2)$. Any element of V is a column vector with two components, the first component belonging to V_1 and the second component belonging to V_2. If c is any primitive idempotent in V, then exactly one component of c is nonzero and this nonzero component is a primitive idempotent in the corresponding component algebra. By rearranging the elements, we may write

$$\{e_1, e_2, \ldots, e_r\} = \left\{\begin{bmatrix} e_{11} \\ 0 \end{bmatrix}, \begin{bmatrix} e_{12} \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} e_{1r_1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e_{21} \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ e_{2r_2} \end{bmatrix}\right\} \quad \text{and}$$

$$\{f_1, f_2, \ldots, f_r\} = \left\{\begin{bmatrix} f_{11} \\ 0 \end{bmatrix}, \begin{bmatrix} f_{12} \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} f_{1r_1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ f_{21} \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ f_{2r_2} \end{bmatrix}\right\}.$$

Note that $\{e_{11}, e_{12}, \ldots, e_{1r_1}\}$ and $\{f_{11}, f_{12}, \ldots, f_{1r_1}\}$ are Jordan frames in V_1 and $\{e_{21}, e_{22}, \ldots, e_{2r_2}\}$ and $\{f_{21}, f_{22}, \ldots, f_{2r_2}\}$ are Jordan frames in V_2. By Theorem IV.2.5 of Faraut and Korányi [6], there exists $\Lambda_i \in \mathrm{Aut}(V_i)$ such that $\Lambda_1(e_{1i}) = f_{1i}$ for all i and $\Lambda_2(e_{2i}) = f_{2i}$ for all i. Clearly, $\Lambda = \Lambda_1 \times \Lambda_2$ belongs to $\mathrm{Aut}(V)$ and $\{\Lambda(e_1), \Lambda(e_2), \ldots, \Lambda(e_r)\} = \{f_1, f_2, \ldots, f_r\}$. □

5. Automorphism invariance.

Given a linear transformation $L: V \to V$, $\Lambda \in \mathrm{Aut}(V)$ and $\Gamma \in \mathrm{Aut}(K)$, we define transformations $\widetilde{L}$ and $\widehat{L}$ by

$$\widetilde{L} := \Lambda^T L \Lambda \qquad \text{and} \qquad \widehat{L} := \Gamma^T L \Gamma.$$

In what follows, we say that a property "T" is invariant under the automorphisms of the algebra (cone) if $\widetilde{L}$ (respectively, $\widehat{L}$) has property T whenever L has property T.

Theorem 5.1. *The following statements hold in V:*
(a) *The* $\mathbf{R}_0$, **Q**, **GUS**, *and Lipschitzian* **GUS**-*properties are invariant under cone automorphisms.*
(b) *The Jordan* **P**, **P**, *and positive principal minor properties are invariant under algebra automorphisms.*

Proof. Fix $L: V \to V$ and $\Gamma \in \mathrm{Aut}(K)$. For any $q \in V$, we claim that

$$\Gamma^{-1}(\mathrm{SOL}(L, q)) = \mathrm{SOL}(\widehat{L}, \Gamma^T q). \tag{5}$$

To see this, let $x \in \mathrm{SOL}(L, q)$ and $y = L(x) + q$. Letting $u := \Gamma^{-1}(x)$, we see that $\Gamma^T(y) = \Gamma^T L\Gamma(u) + \Gamma^T q$. Because Γ and Γ^{-1} preserve K, we have $\Gamma^{-1}(x) \geq 0$, $\Gamma^T(y) \geq 0$, and $\langle \Gamma^T y, \Gamma^{-1} x\rangle = \langle y, x\rangle = 0$. This proves that $\Gamma^{-1}(x) \in \mathrm{SOL}(\widehat{L}, \Gamma^T q)$. By a similar argument, we can show that $\Gamma(\mathrm{SOL}(\widehat{L}, \Gamma^T q)) \subseteq \mathrm{SOL}(L, q)$. This proves the claim.

Now, item (a) easily follows from the equality of the above solution sets. Just to illustrate, suppose that L has the Lipschitzian **GUS**-property. Then, the solution map $q \mapsto \mathrm{SOL}(L, q)$ is single valued and Lipschitzian on V. Because Γ^{-1} is linear, the map $q \mapsto \Gamma^{-1}(\mathrm{SOL}(L, q)) = \mathrm{SOL}(\widehat{L}, \Gamma^T q)$ is also single valued and Lipschitzian on V, proving the Lipschitzian **GUS**-property of $\widehat{L}$.

To prove (b), suppose first that L has the Jordan **P**-property. Let $x \circ \Lambda^T L\Lambda(x) \leq 0$, where $\Lambda \in \mathrm{Aut}(V)$. Take any $z \geq 0$. Then,

$$0 \geq \langle x \circ \Lambda^T L\Lambda(x), z\rangle = \langle \Lambda^T(L(\Lambda(x))), x \circ z\rangle = \langle L(\Lambda(x)), \Lambda(x) \circ \Lambda(z)\rangle = \langle \Lambda(x) \circ L(\Lambda(x)), \Lambda(z)\rangle.$$

Because z is arbitrary in K and $\Lambda(K) = K$, we have $\Lambda(x) \circ L(\Lambda(x)) \leq 0$. Using the Jordan **P**-property of L, we get $\Lambda(x) = 0$ and hence $x = 0$. This proves that $\widetilde{L}$ has the Jordan **P**-property.

Now suppose that L has the **P**-property and let $x \circ \widetilde{L}(x) \leq 0$, where x and $\widetilde{L}(x)$ operator commute. Then, there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ with respect to which x and $\widetilde{L}(x)$ have their spectral representations,

$$x = \sum_1^r \lambda_i e_i \qquad \text{and} \qquad \widetilde{L}(x) = \Lambda^T L\Lambda(x) = \sum_1^r \mu_i e_i.$$

From $x \circ \widetilde{L}(x) \leq 0$, we get $\lambda_i \mu_i \leq 0$ for all i. We now claim that $\Lambda(x)$ and $L(\Lambda(x))$ operator commute and $\Lambda(x) \circ L(\Lambda(x)) \leq 0$. To see this, observe that $\Lambda(x) = \sum_1^r \lambda_i \Lambda(e_i)$ and $L\Lambda(x) = \sum_1^r \mu_i (\Lambda^T)^{-1}(e_i) = \sum_1^r \mu_i \theta_i \Lambda(e_i)$ from item (2) in Proposition 4.2. Because $\{\Lambda(e_1), \Lambda(e_2), \ldots, \Lambda(e_r)\}$ is a Jordan frame, for all i and j, $\Lambda(e_i)$ operator commutes with $\Lambda(e_j)$. We see that $\Lambda(x)$ and $L(\Lambda(x))$ operator commute. Moreover,

$$\Lambda(x) \circ L(\Lambda(x)) = \sum_1^r \lambda_i \mu_i \theta_i \Lambda(e_i) \leq 0.$$

Now, we use the **P**-property of L to conclude that $\Lambda(x) = 0$ and $x = 0$. This proves that $\widetilde{L}$ has the **P**-property.

Now, suppose that L has the positive principal minor property. Then, for any $\Lambda \in \mathrm{Aut}(V)$ and any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we have from Proposition 4.3,

$$(\Lambda^T L\Lambda)_{\{e_1, e_2, \ldots, e_l\}} = \Lambda^T L_{\{f_1, f_2, \ldots, f_l\}} \Lambda.$$

Because the determinant of $L_{\{f_1, f_2, \ldots, f_l\}}$ is positive by assumption, we conclude, by simple linear algebra arguments, that the determinant of $\Lambda^T L_{\{f_1, f_2, \ldots, f_l\}} \Lambda$ $(=(\Lambda^T L\Lambda)_{\{e_1, e_2, \ldots, e_l\}})$ is positive. Hence, $\widetilde{L}$ has the positive principal minor property. This proves (b). ☐

Remark 5.1. (a) By modifying the proof of (b) in the above theorem, we can get the following: Suppose that $\Lambda \in \mathrm{Aut}(V)$ and $\Lambda(V^{(l)}) = W^{(l)}$ as in Proposition 4.3. If $L_{\{f_1, f_2, \ldots, f_l\}}$ has the **P**-property on $W^{(l)}$, then $\Lambda^T L_{\{f_1, f_2, \ldots, f_l\}} \Lambda$ has the **P**-property on $V^{(l)}$.

(b) In Gowda et al. [10], the order **P**-property of a linear transformation $L: V \to V$ is defined by the condition

$$x \sqcap L(x) \leq 0 \leq x \sqcup L(x) \quad \Rightarrow \quad x = 0,$$

where $x \sqcap y := x - (x - y)^+$ and $x \sqcup y := y + (x - y)^+$. It is known that order $\mathbf{P} \Rightarrow$ Jordan $\mathbf{P} \Rightarrow \mathbf{P}$. While the Jordan **P** and **P**-properties are invariant under algebra automorphisms, we do not know if the same holds for order **P**.

Example 5.1. Let $V = \mathcal{S}^2$ and $K = \mathcal{S}^2_+$. Consider the matrices

$$A = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad Z = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, A is positive stable (that is, the real part of every eigenvalue of A is positive), and hence the Lyapunov transformation defined by

$$L_A(X) := AX + XA^T$$

has the Jordan **P**-property (see Gowda and Song [7]). Consider the automorphism $\Gamma \in \operatorname{Aut}(K)$ defined by $\Gamma(X) := QXQ^T$ (see Example 4.1). Then,

$$\widehat{L}_A(X) = Q^T L_A(QXQ^T)Q.$$

By direct computation, it is easily verified that $Z \circ \widehat{L}_A(Z) \le 0$. Hence, $\widehat{L}_A$ does not have the Jordan **P**-property. We conclude that the Jordan **P**-property is not invariant under cone automorphisms.

PROPOSITION 5.1. *Suppose that in V, x_i operator commutes with y_i for $i = 1, 2$, and that $x_1 + x_2$ operator commutes with $y_1 + y_2$. Then, $x_1 - x_2$ operator commutes with $y_1 - y_2$.*

PROOF. We have $L_{x_i} L_{y_i} = L_{y_i} L_{x_i}$ for $i = 1, 2$, and $L_{x_1+x_2} L_{y_1+y_2} = L_{y_1+y_2} L_{x_1+x_2}$. Using these in the equation

$$L_{x_1-x_2} L_{y_1-y_2} = (L_{x_1} - L_{x_2})(L_{y_1} - L_{y_2}) = 2[L_{x_1} L_{y_1} + L_{x_2} L_{y_2}] - L_{x_1+x_2} L_{y_1+y_2},$$

we get $L_{x_1-x_2} L_{y_1-y_2} = L_{y_1-y_2} L_{x_1-x_2}$, proving the assertion. □

PROPOSITION 5.2. *Let $L: V \to V$ be linear such that for all $\Gamma \in \operatorname{Aut}(K)$, $\widehat{L} = \Gamma^T L \Gamma$ has the* **P***-property. Suppose that for a $q \in V$, x_1 and x_2 are solutions of* LCP(L, q) *such that $x_1 + x_2 > 0$. Then, $x_1 = x_2$.*

PROOF. By taking the identity transformation for Γ, we see that L has the **P**-property. First, we consider the case when $x_1 + x_2 = e$, where e is the unit element in V. Let $y_i := L(x_i) + q$. Then, $x_i \ge 0$, $y_i \ge 0$, and $x_i \circ y_i = 0$ for $i = 1, 2$. From Proposition 2.1, x_i operator commutes with y_i $(i = 1, 2)$. Also, $x_1 + x_2 = e$ operator commutes with $y_1 + y_2$. By the previous proposition, $x_1 - x_2$ operator commutes with $y_1 - y_2 = L(x_1 - x_2)$. Also,

$$\begin{aligned}(x_1 - x_2) \circ L(x_1 - x_2) &= (x_1 - x_2) \circ (y_1 - y_2) = -(x_1 \circ y_2 + x_2 \circ y_1)\\ &= -(x_1 + x_2) \circ (y_1 + y_2) = -(y_1 + y_2) \le 0.\end{aligned}$$

Now, by the **P**-property of L, we get $x_1 = x_2$.

We now consider the general case. Because $x_1 + x_2 > 0$ and K is a symmetric cone, there exists a $\Gamma \in \operatorname{Aut}(K)$ such that $\Gamma^{-1}(x_1 + x_2) = e$. Letting $u_i = \Gamma^{-1}(x_i)$, $y_i := L(x_i) + q$, and $v_i = \Gamma^T y_i$ for $i = 1, 2$, we have $u_i \in \operatorname{SOL}(\widehat{L}, \Gamma^T(q))$ by Equation (5). Because $u_1 + u_2 = e$, by the previous case (applied to $\widehat{L}$), we get $u_1 = u_2$, and hence $x_1 = x_2$. □

COROLLARY 5.1. *Consider a linear transformation $L: V \to V$ such that for all $\Gamma \in \operatorname{Aut}(K)$, $\widehat{L} = \Gamma^T L \Gamma$ has the* **P***-property. Then, the system*

$$x > 0 \qquad \text{and} \qquad L(x) \le 0$$

cannot have a solution in V.

PROOF. Suppose that there is $x_1 > 0$ such that $L(x_1) \le 0$. Define $q := -L(x_1)$. As $q \ge 0$, x_1 and $x_2 := 0$ are two solutions of LCP(L, q) with the property that $x_1 + x_2 > 0$. This cannot happen in view of the previous proposition. □

EXAMPLE 5.2. Consider four real numbers a, b, c, d with $a > 0$, $d > 0$, and $b \ne 0$. On $\mathcal{S}^2$, define L as follows:

$$\text{for any } X = \begin{bmatrix} x & y \\ y & z \end{bmatrix}, \qquad L(X) = \begin{bmatrix} ax + by & -bx \\ -bx & cy + dz \end{bmatrix}.$$

It can be easily seen that L has the **P**-property. If we further assume that

$$b^2 > 4ad, \qquad \frac{2c}{b} > 1 + \frac{b^2}{a^2},$$

then

$$Z := \begin{bmatrix} 1/a & -2/b \\ -2/b & 1/d \end{bmatrix} > 0 \qquad \text{and} \qquad L(Z) = \begin{bmatrix} -1 & -b/a \\ -b/a & 1 - 2c/b \end{bmatrix} \le 0.$$

In view of the above corollary, there exists some $\Gamma \in \operatorname{Aut}(K)$, and $\widehat{L} = \Gamma^T L \Gamma$ does not have the **P**-property. This example shows that the **P**-property is not invariant under cone automorphisms.

6. Ultra P- and super P-properties. It is well known that any principal submatrix of a **P**-matrix is itself a **P**-matrix. However, in the context of a general Euclidean Jordan algebra, the **P**- and **GUS**-properties are not inherited by principal substransformations, see e.g., Gowda et al. [9, p. 362]. In the context of $\mathcal{S}^n$, Gowda and Song [7] introduced, as a sufficient condition for the **GUS**-property, the so-called $\mathbf{P}_2$-property of a linear transformation L: $\mathcal{S}^n \to \mathcal{S}^n$ by the implication

$$X \geq 0,\ Y \geq 0,\ (X-Y)L(X-Y)(X+Y) \leq 0 \quad \Rightarrow \quad X = Y.$$

It was shown in Gowda et al. [9] that this property is equivalent to saying that for every invertible $Q \in R^{n\times n}$, every principal subtransformation of $\widehat{L}$ has the **P** (equivalently, **GUS**)-property, where

$$\widehat{L}(X) := Q^T L(QXQ^T)Q \quad (X \in \mathcal{S}^n).$$

This $\mathbf{P}_2$-property, which involves the ordinary (associative) product of three square matrices, cannot be extended to a general Euclidean Jordan algebra as there may not be an associative (triple) product in a general setting. However, we use its equivalent formulation to introduce the so-called ultra **P**-property (a term used in Gowda et al. [8]) of a linear transformation by demanding that for every $\Gamma \in \mathrm{Aut}(K)$, every principal subtransformation of $\widehat{L} = \Gamma^T L \Gamma$ should have the **P**-property. We then show that the ultra **P**-property implies the **GUS**-property.

By replacing the cone automorphisms in the definition of the ultra **P**-property by algebra automorphisms, we get the definition of super **P**-property. As we shall see, this property is equivalent to saying that every principal subtransformation of L has the **P**-property. On $\mathcal{S}^n$, the super **P**-property was defined and studied in Gowda et al. [8] using principal subtransformations and algebra automorphisms (of $\mathcal{S}^n$). Malik and Mohan [15] characterize this super **P**-property by considering the faces of the semidefinite cone $\mathcal{S}^n_+$: For every nonzero face F of $\mathcal{S}^n_+$, the transformation $L_{FF} := P_{\mathrm{span}(F)} L$: $\mathrm{span}(F) \to \mathrm{span}(F)$ has the **P**-property. In Theorem 6.3 below, we extend this characterization to arbitrary Euclidean Jordan algebras.

Definition 6.1. Consider a linear transformation L: $V \to V$. We say that L has

(i) The super **P**-property (super **GUS**-property) if for any $\Lambda \in \mathrm{Aut}(V)$, every principal subtransformation of $\widetilde{L} = \Lambda^T L \Lambda$ has the **P**-property (respectively, **GUS**-property);

(ii) The ultra **P**-property (ultra **GUS**-property) if for any $\Gamma \in \mathrm{Aut}(K)$, every principal subtransformation of $\widehat{L} = \Gamma^T L \Gamma$ has the **P**-property (respectively, **GUS**-property).

Here is our first observation.

Theorem 6.1. *For any linear mapping L: $V \to V$,*

$$\textit{strong monotonicity} \Rightarrow \textit{ultra } \mathbf{P} \Rightarrow \textit{super } \mathbf{P} \Rightarrow \mathbf{P}, \quad \textit{and}$$

$$\textit{strong monotonicity} \Rightarrow \textit{ultra } \mathbf{GUS} \Rightarrow \textit{super } \mathbf{GUS} \Rightarrow \mathbf{GUS}.$$

Moreover, the ultra **P**- *and ultra* **GUS**-*properties are invariant under cone automorphisms, and the super* **P**- *and super* **GUS**-*properties are invariant under algebra automorphisms.*

Proof. We (only) show that strong monotonicity implies the ultra **GUS**-property. Assume that L is strongly monotone. It follows that $\widehat{L} = \Gamma^T L \Gamma$ is strongly monotone. In fact,

$$\langle \widehat{L}(x), x\rangle = \langle \Gamma^T L \Gamma(x), x\rangle = \langle L\Gamma(x), \Gamma(x)\rangle > 0 \quad \text{for } x \neq 0.$$

Similarly, for any Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V and any $1 \leq l \leq r$, $\widehat{L}_{\{e_1, e_2, \ldots, e_l\}}$ is strongly monotone. Then, $\widehat{L}_{\{e_1, e_2, \ldots, e_l\}}$ has the **GUS**-property (Gowda et al. [10]), which in turn, shows that L has the ultra **GUS**-property. ☐

We now relate the ultra **P**-property to the **GUS**-property.

Theorem 6.2. *For any linear transformation L: $V \to V$,*

$$\textit{ultra } \mathbf{P} \Rightarrow \mathbf{GUS}.$$

Proof. Assume that L has the ultra **P**-property. Fix $q \in V$, and let x_1 and x_2 be two solutions of LCP(L, q).

Case 1. $x_1 + x_2 \in \mathrm{int}\, K$. In this case, $x_1 = x_2$ by Proposition 5.2.

Case 2. $x_1 + x_2 \in \partial K$. Without loss of generality, assume that $x_1 + x_2 \neq 0$, otherwise $x_1 = x_2 = 0$. Let $x_1 + x_2 = \sum_{i=1}^{l} \lambda_i e_i + 0 \cdot e_{l+1} + \cdots + 0 \cdot e_r$ $(1 \leq l < r)$ and $\lambda_i > 0$ for $1 \leq i \leq l$, where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame corresponding to $x_1 + x_2$. If we let $a = \sum_{i=1}^{l} \sqrt{\lambda_i} e_i + e_{l+1} + \cdots + e_r$, then a is invertible and $\Gamma := P_a \in \mathrm{Aut}(K)$ by Proposition III.2.2 in Faraut and Korányi [6], where $P_a(x) = 2a \circ (a \circ x) - a^2 \circ x$ for any x. We easily verify that $(x_1 + x_2) = \Gamma(e_1 + \cdots + e_l)$. Now, put $u_i := \Gamma^{-1}(x_i)$, $y_i = L(x_i) + q$, and $z_i := \Gamma^T y_i = \widehat{L}(u_i) + \Gamma^T q$. Then, u_1 and u_2 are solutions of LCP$(\widehat{L}, \Gamma^T q)$. Now, $u_1 + u_2 = e_1 + \cdots + e_l \in V^{(l)} = \{x \in V\colon x \circ (e_1 + \cdots + e_l) = x\}$ and $u_1, u_2 \geq 0$. By Proposition 3.3, $u_1, u_2 \in V^{(l)}$. We have

$$(u_1 - u_2) \circ \widehat{L}_{\{e_1, \ldots, e_l\}}(u_1 - u_2) = (u_1 - u_2) \circ P^{(l)} \widehat{L}(u_1 - u_2) = (u_1 - u_2) \circ (v_1 - v_2), \tag{6}$$

where $v_i = P^{(l)}(z_i)$. By Remark 4.2(b), $v_i \geq 0$ in $V^{(l)}$. Because $u_i \geq 0$ and

$$\langle v_i, u_i \rangle = \langle P^{(l)}(z_i), u_i \rangle = \langle z_i, P^{(l)}(u_i) \rangle = \langle z_i, u_i \rangle = 0,$$

we have $u_i \circ v_i = 0$ for $i = 1, 2$, from Proposition 2.1. Thus, Equation (6) continues as follows:

$$(u_1 - u_2) \circ (v_1 - v_2) = -(u_1 + u_2) \circ (v_1 + v_2) = -(v_1 + v_2) \leq 0 \quad (\text{in } V^{(l)}).$$

Here we have used the facts that $v_1 + v_2 \in V^{(l)}$ and $e_1 + \cdots + e_l$ is the identity in $V^{(l)}$. By applying Proposition 5.1, we see that $(u_1 - u_2)$ and $(v_1 - v_2)$ operator commute in $V^{(l)}$. Now, using the **P**-property of $\widehat{L}_{\{e_1, \ldots, e_l\}}$, we get $u_1 = u_2$. This, in turn, implies that $x_1 = x_2$, and concludes the proof of Theorem 6.2. □

Remark 6.1. (a) It has been shown in Gowda et al. [8] that for $A \in R^{n \times n}$, the Lyapunov transformation L_A on $\mathcal{S}^n$ defined by $L_A(X) = AX + XA^T$ has the ultra/super **P**-(**GUS**)-property if and only if A is positive definite. Because L_A has the **GUS**-property if and only if A is positively stable and positive semidefinite, we see that the converse in the above theorem does not hold.

(b) It has been proved in Gowda et al. [9] that for $A \in R^{n \times n}$, the transformation M_A on $\mathcal{S}^n$ defined by $M_A(X) = AXA^T$ has the ultra/super **P**-(**GUS**)-property if and only if A is either positive definite or negative definite.

(c) We strongly suspect that the ultra **GUS**- and ultra **P**-properties are the same and that the ultra **P**- and super **P**-properties are different. However, we have neither proofs nor examples to justify these statements.

We know from Remark 6.1(a) that the converse in the previous theorem does not hold. What if L has the Lipschitzian **GUS**-property? Generalizing some recent results of Balaji et al. [1] (proved in the setting of $\mathcal{S}^n$), we show that the Lipschitzian **GUS**-property implies the ultra **P**-property under certain conditions.

Proposition 6.1. *Under each of the following conditions,*

Lipschitzian **GUS** $\Rightarrow$ *ultra* **GUS**.

(a) *The rank of V is* 2.
(b) *L is monotone.*

Proof. Assume that L has the Lipschitzian **GUS**-property. Because the Lipschitzian **GUS**-property is invariant under automorphisms of the cone, to get the required implication, we show that any principal subtransformation of L has the **GUS**-property.

Suppose that condition (a) holds, and let $\{e_1, e_2\}$ be any Jordan frame in V. Because $e_1 + e_2 = e$ (the unit element in V), the eigenspace corresponding to $e_1 + e_2$ is V. That L restricted to this eigenspace has the **GUS**-property follows from the Lipschitzian **GUS**-property of L. Moreover, the principal subtransformation $L_{\{e_1\}}$ defined on the eigenspace of e_1 (which is Re_1) has a positive determinant, as the Lipschitzian **GUS**-property implies the positive principal minor property (see Gowda et al. [10]). On this one-dimensional eigenspace, $L_{\{e_1\}}$ has the **GUS**-property. We conclude that every principal subtransformation of L has the **GUS**-property.

Now, suppose that condition (b) holds. Consider any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, and the principal subtransformation $S := L_{\{e_1, e_2, \ldots, e_l\}}\colon V^{(l)} \to V^{(l)}$. Suppose that a nonzero element x in $V^{(l)}$ operator commutes with $S(x)$ and $x \circ S(x) \leq 0$. Because L is monotone, it is easily seen that S is also monotone, and hence $0 \geq \langle x \circ S(x), e \rangle = \langle x, S(x) \rangle \geq 0$. It follows that $\langle x \circ S(x), e \rangle = 0$. By Proposition 2.1, $x \circ S(x) = 0$. As x and $S(x)$ operator commute, we may write

$$x = \sum_{i=1}^{l} \lambda_i f_i \qquad \text{and} \qquad S(x) = \sum_{i=1}^{l} \mu_i f_i,$$

where $\{f_1, f_2, \ldots, f_l\}$ is a Jordan frame in $V^{(l)}$. Because of $x \circ S(x) = 0$, we have $\lambda_i \mu_i = 0$ for all i. Thus, we may write $x = \sum_{i=1}^{k} \lambda_i f_i$ and $S(x) = \sum_{i=k+1}^{l} \mu_i f_i$ for some k between 1 and l. If $Q^{(k)}$ denotes the projection transformation from V onto the eigenspace $W^{(k)}$ of $f_1+f_2+\cdots+f_k$, then $0=Q^{(k)}(S(x))=Q^{(k)}L_{\{e_1,e_2,\ldots,e_l\}}(x)=Q^{(k)}P^{(l)}L(x)$. Because the range of $Q^{(k)}$ is contained in the range of $P^{(l)}$, we have $Q^{(k)}P^{(l)}=Q^{(k)}$. Hence, $0=Q^{(k)}L(x)$. This means that the principal subtransformation $Q^{(k)}L\colon W^{(k)}\to W^{(k)}$ is not invertible, contradicting the positive principal minor property. Thus, $L_{\{e_1,e_2,\ldots,e_l\}}$ has the **P**-property. Now, the **GUS**-property follows from the fact that monotonicity along with the **P**-property implies the **GUS**-property (see Gowda et al. [10]). □

Remark 6.2. We note that any Euclidean Jordan algebra of rank 2 is isomorphic to $\mathcal{L}^n$; see Corollary IV.1.5 in Faraut and Korányi [6]. In particular, $\mathcal{S}^2$ is isomorphic to $\mathcal{L}^3$. This latter statement can be seen directly by considering the algebra isomorphism $A\colon \mathcal{S}^2\to\mathcal{L}^3$,

$$A\left(\begin{bmatrix} x & y\\ y & z\end{bmatrix}\right)=\left[\frac{x+z}{2},\,\frac{x-z}{2},\,y\right]^T.$$

The following theorem generalizes Proposition 2 in Malik and Mohan [15].

Theorem 6.3. *For a linear transformation $L\colon V\to V$, the following are equivalent*:

(1) *For every nonzero face F of K, the transformation $L_{FF}:=P_{\operatorname{span}(F)}L\colon \operatorname{span}(F)\to\operatorname{span}(F)$ has the* **P**-*property, i.e.,*

$$\left.\begin{array}{r} x\in\operatorname{span}(F),\ x \text{ and } L_{FF}(x) \text{ operator commute}\\ x\circ L_{FF}(x)\le 0\end{array}\right\}\Rightarrow x=0.$$

(2) *For any Jordan frame $\{e_1,e_2,\ldots,e_r\}$ in V, for any $1\le l\le r$, the linear transformation $L_{\{e_1,e_2,\ldots,e_l\}}$ has the* **P**-*property.*

(3) *Let $\{e_1^0,e_2^0,\ldots,e_r^0\}$ be any (fixed) Jordan frame in V. Then, for all $\Lambda\in\operatorname{Aut}(V)$, for all $1\le l\le r$, and for all $\{k_1,k_2,\ldots,k_l\}\subseteq\{1,2,\ldots,r\}$, the transformation $(\Lambda^TL\Lambda)_{\{e^0_{k_1},e^0_{k_2},\ldots,e^0_{k_l}\}}$ has the* **P**-*property.*

(4) *L has the super* **P**-*property.*

Proof. In view of Theorem 3.1, every nonzero face F is given by $K^{(l)}$ and $\operatorname{span}(F)=V^{(l)}$ for an appropriate Jordan frame and l. So, the transformation L_{FF} is nothing but a principal subtransformation of L. Thus, (1) and (2) are equivalent.

Now, fix any Jordan frame $\{e_1^0,e_2^0,\ldots,e_r^0\}$ in V and take any $\Lambda\in\operatorname{Aut}(V)$. Let $\Lambda(e_i^0)=e_i$ for all i. Then, $(\Lambda^{-1})^T(\Lambda^TL\Lambda)_{\{e^0_{k_1},e^0_{k_2},\ldots,e^0_{k_l}\}}\Lambda^{-1}=L_{\{e_{k_1},e_{k_2},\ldots,e_{k_l}\}}$ by Proposition 4.3. Now, we apply Remark 5.1(a) to get the implication (2) $\Rightarrow$ (3).

Now, assume (3) and let $\{e_1,e_2,\ldots,e_r\}$ be any Jordan frame and $\Lambda\in\operatorname{Aut}(V)$. By Proposition 4.4, there exists a $\Lambda_0\in\operatorname{Aut}(V)$ and a permutation σ of $\{1,2,\ldots,r\}$ such that $\Lambda_0(e^0_{\sigma(i)})=e_i$ for all i. By Proposition 4.3,

$$(\Lambda^TL\Lambda)_{\{e_1,e_2,\ldots,e_l\}}=(\Lambda_0^{-1})^T(\Lambda_0^T\Lambda^TL\Lambda\Lambda_0)_{\{e^0_{\sigma(1)},e^0_{\sigma(2)},\ldots,e^0_{\sigma(l)}\}}\Lambda_0^{-1}.$$

Because $\Lambda\Lambda_0\in\operatorname{Aut}(V)$, by our assumption $(\Lambda_0^T\Lambda^TL\Lambda\Lambda_0)_{\{e^0_{\sigma(1)},e^0_{\sigma(2)},\ldots,e^0_{\sigma(l)}\}}$ has the **P**-property on $V_0^{(l)}$ (the eigenspace of $e^0_{\sigma(1)}+e^0_{\sigma(2)}+\cdots+e^0_{\sigma(l)}$). From Remark 5.1(a), $(\Lambda^TL\Lambda)_{\{e_1,e_2,\ldots,e_l\}}$ has the **P**-property on $V^{(l)}$. Thus, we have (4).

To see (4) $\Rightarrow$ (2), we take $\Lambda=I$ in the definition of the super **P**-property. □

Proposition 6.2. *Super* **GUS**-*property $\Rightarrow$ super* **P**-*property $\Rightarrow$ positive principal minor property. Converse implications hold when L is monotone.*

Proof. Clearly, the super **GUS**-property implies the super **P**-property. That the super **P**-property implies the positive principal minor property follows from the fact that the determinant of a transformation with the **P**-property is positive (see Gowda et al. [10]). To see the converse, assume that L is monotone. Then, we proceed as in the proof of part (b) in Proposition 6.1 to show that positive principal minor property implies the super **GUS**-property. □

Concluding remarks. In this paper, we studied the invariance of some complementarity properties under (Euclidean Jordan) algebra and (symmetric) cone automorphisms. We also studied ultra and super **P**-(**GUS**)-properties for linear transformations. Known implications between various properties are shown in Figure 1.

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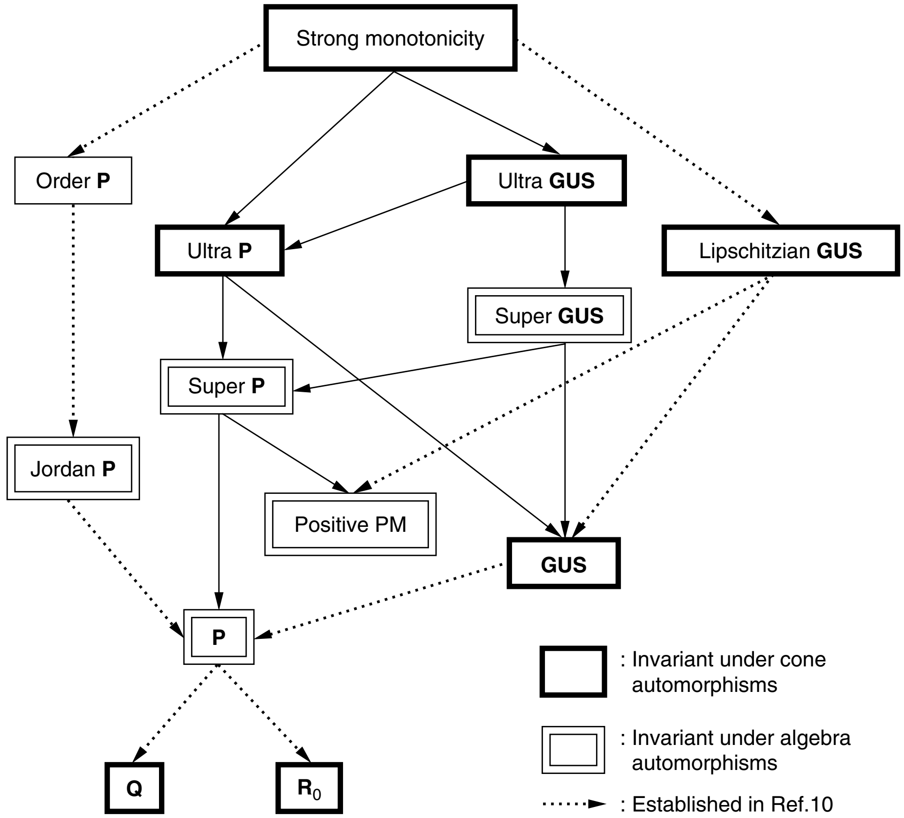

FIGURE 1. Interrelations between various **P**- and **GUS**-properties.

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