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# The automorphism group of a completely positive cone and its Lie algebra

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## ABSTRACT

Given a closed cone  $\mathcal{C}$  in  $R^n$ , we consider the corresponding completely positive (convex) cone  $\mathcal{K}$  generated by  $\{uu^T : u \in \mathcal{C}\}$  in  $S^n$ . Under certain conditions on  $\mathcal{C}$ , we describe the automorphism group of  $\mathcal{K}$  and its corresponding Lie algebra in terms of those of  $\mathcal{C} \cup -\mathcal{C}$  and/or  $\mathcal{C}$ . In particular, we show that when  $\mathcal{C}$  is a (closed convex) proper cone, the automorphism groups of  $\mathcal{C}$  and  $\mathcal{K}$  are isomorphic and their corresponding Lie algebras are isomorphic.

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## 1. Introduction

Given a closed cone  $\mathcal{C}$  in  $R^n$  that is not necessarily convex, we consider the corresponding *completely positive cone*  $\mathcal{K}$  which is the (closed) convex cone generated by  $\{uu^T : u \in \mathcal{C}\}$  in the space  $S^n$  of all  $n \times n$  real symmetric matrices. This paper deals with the automorphism group of  $\mathcal{K}$  and its Lie algebra.

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To motivate our discussion, let  $C = R^n$ . In this case, the corresponding completely positive cone is  $S_+^n$ , the cone of all positive semidefinite matrices in  $S^n$ . For this cone, the automorphism group  $\text{Aut}(S_+^n)$  (which consists of invertible linear transformations on  $S^n$  mapping  $S_+^n$  onto itself) and its Lie algebra  $\text{Lie}(\text{Aut}(S_+^n))$  are known (see [10] or [8], Theorem 13, pp. 150 and [4] or [7], Example 1):

- Every element  $L$  in  $\text{Aut}(S_+^n)$  is of the form  $\widehat{Q}$ , where  $Q$  is an invertible matrix on  $R^n$  and

$$\widehat{Q}(X) := QXQ^T \text{ for all } X \in S^n.$$

- Every element of  $\text{Lie}(\text{Aut}(S_+^n))$  is of the form  $L_A$  for some matrix  $A \in R^{n \times n}$ , where  $L_A$  is defined on  $S^n$  by

$$L_A(X) := AX + XA^T.$$

Note that in the above statements,  $Q$  (which is invertible) belongs to  $\text{Aut}(R^n)$ , the automorphism group of the cone  $R^n$ , and  $A$  belongs to  $R^{n \times n}$ , the Lie algebra of  $\text{Aut}(R^n)$ . This means that the automorphism group of  $S_+^n$  and its Lie algebra can be described by those of the underlying cone, namely,  $R^n$ . This raises the question whether an analogous result holds for other closed cones. In this article, we prove the following.

**Theorem 1.** *Let  $C$  be a closed cone in  $R^n$ ,  $\widetilde{C} := C \cup -C$ , and  $\mathcal{K}$  be the completely positive cone of  $C$ . Let  $\text{Aut}(\widetilde{C})$  and  $\text{Aut}(\mathcal{K})$  denote, respectively, the automorphism groups of  $\widetilde{C}$  and  $\mathcal{K}$  in  $S^n$ . Suppose that  $C$  has nonempty interior. Then the following hold:*

- The mapping  $Q \mapsto \widehat{Q} : \text{Aut}(\widetilde{C}) \rightarrow \text{Aut}(\mathcal{K})$  is a two-to-one surjective group homomorphism.*
- The mapping  $A \mapsto L_A : \text{Lie}(\text{Aut}(\widetilde{C})) \rightarrow \text{Lie}(\text{Aut}(\mathcal{K}))$  is a Lie algebra isomorphism.*

**Theorem 2.** *Suppose  $C$  is a closed pointed cone in  $R^n$  with nonempty interior. Then,*

- the mapping  $A \mapsto L_A : \text{Lie}(\text{Aut}(C)) \rightarrow \text{Lie}(\text{Aut}(\mathcal{K}))$  is a Lie algebra isomorphism.*
- If, in addition,  $C \setminus \{0\}$  is also connected, then*
- the mapping  $Q \mapsto \widehat{Q} : \text{Aut}(C) \rightarrow \text{Aut}(\mathcal{K})$  is a group isomorphism.*

*In particular, conclusions (i) and (ii) hold when  $C$  is a proper cone, that is,  $C$  is a closed convex pointed cone with nonempty interior.*

In Theorem 2, Item (i) is a refinement of a recent result of [6] which says that for a proper cone  $C$ , the mapping in Item (i) is injective.

## 2. Preliminaries

Throughout this paper,  $(H, \langle \cdot, \cdot \rangle)$  denotes a real finite dimensional inner product space. Let  $K$  denote a closed cone in  $H$ , that is,  $K$  is closed in  $H$  and for  $x \in K, \lambda \geq 0$  in  $R$  we have  $\lambda x \in K$ . For such a cone, its interior is denoted by  $\text{int}(K)$  and its dual is given by  $K^* := \{y \in H : \langle y, x \rangle \geq 0 \ \forall x \in K\}$ . Following [2], we say that a closed cone  $K$  is

- *pointed* if  $K \cap -K = \{0\}$ ;
- *proper* if  $K$  is a closed convex pointed cone with nonempty interior;

Given a linear transformation  $L : H \rightarrow H$  and a closed cone  $K$  in  $H$ , we say that

- $L$  is *copositive* on  $K$  if  $\langle L(x), x \rangle \geq 0$  for all  $x \in K$ ;
- $L$  is *Lyapunov-like* on  $K$  if  $K \ni x \perp y \in K^* \Rightarrow \langle L(x), y \rangle = 0$ ;
- $L$  is an *automorphism* of  $K$  if  $L$  is invertible and  $L(K) = K$ .

We denote the group of all automorphisms of  $K$  by  $\text{Aut}(K)$ ; we say that two objects of  $\text{Aut}(K)$  are equal if they take identical values on the entire space  $H$ . We denote the Lie algebra of  $\text{Aut}(K)$  by  $\text{Lie}(\text{Aut}(K))$ . Recall that  $L \in \text{Lie}(\text{Aut}(K))$  if there is a differentiable curve  $Q(t) : (-\delta, \delta) \rightarrow \text{Aut}(K)$  such that  $Q(0) = \text{Id}$  (Identity transformation) and (derivative)  $Q'(0) = L$ . As  $\text{Aut}(K)$  is a matrix group, its Lie algebra is also given by (see [1, Section 7.6])

$$\text{Lie}(\text{Aut}(K)) := \{L \in \mathcal{B}(H, H) : e^{tL} \in \text{Aut}(K) \text{ for all } t \in \mathbb{R}\},$$

where  $\mathcal{B}(H, H)$  denotes the set of all (bounded) linear transformations on  $H$ .

By using the results of [11], we may state the following.

**Theorem 3.** For any proper cone  $K$ ,

$$L \in \text{Lie}(\text{Aut}(K)) \Leftrightarrow L \text{ is Lyapunov – like on } K.$$

In the space  $H = \mathbb{R}^n$ , vectors are written as column vectors and the usual inner product is written as  $\langle x, y \rangle$  or as  $x^T y$ . We denote the nonnegative orthant by  $\mathbb{R}_+^n$ . The set of all  $n \times n$  real matrices is denoted by  $\mathbb{R}^{n \times n}$ ; throughout,  $I$  denotes the identity matrix in  $\mathbb{R}^{n \times n}$ . A matrix  $A \in \mathbb{R}^{n \times n}$  is said to be positive semidefinite if it is copositive on  $\mathbb{R}^n$ .

The space  $H = \mathcal{S}^n$  consists of all  $n \times n$  real symmetric matrices and carries the trace inner product  $\langle X, Y \rangle = \text{trace}(XY)$ , where the trace of a matrix is the sum of all its diagonal elements. We recall that for any two real matrices  $A$  and  $B$ ,  $\text{trace}(AB) = \text{trace}(BA)$ ; hence when  $B = uu^T$  for a column vector  $u$ , we have  $\text{trace}(AB) = u^T Au$ . We denote the (symmetric) cone of all positive semidefinite matrices in  $\mathcal{S}^n$  by  $\mathcal{S}_+^n$ .

Given  $A \in \mathbb{R}^{n \times n}$ , the corresponding Lyapunov transformation  $L_A$  is defined by

$$L_A(X) = AX + XA^T \quad (X \in \mathcal{S}^n).$$

For any invertible matrix  $Q$  in  $\mathbb{R}^{n \times n}$ , we define the transformation  $\widehat{Q}$  on  $\mathcal{S}^n$  by

$$\widehat{Q}(X) := QXQ^T \quad (X \in \mathcal{S}^n).$$

**Proposition 4.** The following statements hold:

- (i) For  $A, B \in \mathbb{R}^{n \times n}$ ,  $L_A = L_B \Rightarrow A = B$ .
- (ii) For  $Q$  and  $P$  invertible in  $\mathbb{R}^{n \times n}$ ,  $\widehat{Q} = \widehat{P} \Rightarrow Q = \pm P$ .

**Proof**

- (i) Since  $L_A = L_B \Rightarrow L_{A-B} = 0$ , it is enough to show that  $L_A = 0 \Rightarrow A = 0$ . When  $L_A = 0$ , we have  $AX + XA^T = 0$  for any  $X$  in  $\mathcal{S}^n$ , and in particular, for any arbitrary diagonal matrix  $X$ . From this, we easily deduce that  $A = 0$ .
- (ii) Since  $\widehat{Q} = \widehat{P} \Rightarrow \widehat{P}^{-1}\widehat{Q} = I$ , it is enough to show that  $\widehat{Q} = I \Rightarrow Q = \pm I$ . Now  $\widehat{Q} = I \Rightarrow QXQ^T = X$  for all  $X \in \mathcal{S}^n$ . By taking  $X = e_i e_i^T$  and  $X = e_i e_j^T + e_j e_i^T$ , where  $e_1, e_2, \dots, e_n$  are the standard coordinate vectors in  $\mathbb{R}^n$ , we deduce that  $Q = \pm I$ .  $\square$

Throughout this paper,  $C$  denotes a closed cone in  $\mathbb{R}^n$  (which is not necessarily convex) and  $\widetilde{C} := C \cup -C$ . We note that

$$\text{int}(C) \neq \emptyset \Leftrightarrow \text{int}(\widetilde{C}) \neq \emptyset.$$

This can be seen, for example, by an application of the Baire category Theorem: If a closed ball in  $\mathbb{R}^n$  is a union of two closed sets in  $\mathbb{R}^n$ , then one of the sets must have nonempty interior in  $\mathbb{R}^n$ .

(When  $C$  is pointed, the sets  $C \setminus \{0\}$  and  $-(C \setminus \{0\})$  are separated in the sense that each is disjoint from the closure of the other. In this case, a connectedness argument can be used instead of the Baire category Theorem.)

Corresponding to  $\mathcal{C}$ , the completely positive cone  $\mathcal{K}$  and the copositive cone  $\mathcal{E}$  in  $S^n$  are defined, respectively, by

$$\mathcal{K} := \left\{ \sum uu^T : u \in \mathcal{C} \right\} \tag{1}$$

and

$$\mathcal{E} := \{A \in S^n : A \text{ copositive on } \mathcal{C}\}, \tag{2}$$

where  $\sum uu^T$  denotes a finite sum of matrices of the form  $uu^T$ . Note that  $\mathcal{K}$  is unchanged if  $\mathcal{C}$  is replaced by  $\bar{\mathcal{C}}$ . Also, every element in  $\mathcal{K}$  is a matrix of the form  $BB^T$  with columns of  $B$  coming from  $\mathcal{C}$  (equivalently,  $u \in \bar{\mathcal{C}}$ ). When  $\mathcal{C} = R_+^n$ , the objects of  $\mathcal{K}$  are called completely positive matrices [3].

**Proposition 5.** *Given a closed cone  $\mathcal{C}$ , the following statements hold:*

- (i)  $\mathcal{E}$  is a closed convex cone in  $S^n$  and  $\mathcal{K} \subseteq S_+^n \subseteq \mathcal{E}$ .
- (ii)  $\mathcal{K}$  is a closed convex cone and  $\mathcal{E}$  is the dual of  $\mathcal{K}$ .
- (iii) If  $\mathcal{C}$  has nonempty interior, then  $\mathcal{K}$  and  $\mathcal{E}$  are proper cones.

The proof of this proposition is somewhat routine. That  $\mathcal{K}$  is closed in  $S^n$  follows from a standard argument via an application of the Carathéodory Theorem for cones [3]. It is easy to verify that  $\mathcal{K}$  is always pointed. If  $\mathcal{K} - \mathcal{K} \neq S^n$ , then there is some nonzero  $A$  in  $S^n$  which is orthogonal to the subspace  $\mathcal{K} - \mathcal{K}$ ; hence  $u^T A u = 0$  for all  $u \in \mathcal{C}$ . When  $\text{int}(\mathcal{C})$  is nonempty, say,  $d \in \text{int}(\mathcal{C})$ , for any  $x \in R^n$  and small  $\varepsilon > 0$ , we can let  $u = d + \varepsilon x \in \mathcal{C}$  to get  $(d + \varepsilon x)^T A (d + \varepsilon x) = 0$ . This leads to  $x^T A x = 0$  for all  $x$ . Since  $A \in S^n$ , this implies  $Ax = 0$  for all  $x$ ; hence  $A = 0$ . This shows that when  $\text{int}(\mathcal{C}) \neq \emptyset$ ,  $\mathcal{K} - \mathcal{K} = S^n$ . As  $\mathcal{K}$  is convex, it follows that  $\text{int}(\mathcal{K}) \neq \emptyset$ . Finally, when  $\mathcal{K}$  is proper, its dual  $\mathcal{E}$  is also proper, see [2].

**Proposition 6.** *Let  $\mathcal{C}$  be a closed cone. Then the following hold:*

- (i) For  $u \in \mathcal{C}$  and  $x \in R^n$ , we have  $[xx^T = uu^T \Rightarrow x = u \text{ or } x = -u]$ .
- (ii) When  $\mathcal{C}$  is pointed, for  $u, v \in \mathcal{C}$ , we have  $[vv^T = uu^T \Rightarrow v = u]$ .

**Proof**

- (i) Take  $u \in \mathcal{C}$  and  $x \in R^n$  with  $xx^T = uu^T$ . As  $x_i^2 = u_i^2$  for all  $i$ , we may assume that  $x$  and  $u$  are nonzero. In this case,  $xx^T x = uu^T x$  and so  $x = \lambda u$  for some  $\lambda \in R$ . Then  $xx^T = uu^T$  implies that  $\lambda^2 = 1$ . Thus,  $x = u$  or  $x = -u$ .
- (ii) Let  $u, v \in \mathcal{C}$  with  $vv^T = uu^T$ . From (i),  $v = u$  or  $v = -u$ . If  $v = -u$ , then  $v \in \mathcal{C} \cap -\mathcal{C} = \{0\}$  and so  $u = -v = 0$ . When  $v \neq 0$ , we must have  $v = u$ .  $\square$

Recall that for a nonzero element  $x \in \mathcal{K}$ , the ray  $\{\lambda x : \lambda \geq 0\} \subseteq \mathcal{K}$  is an extreme ray of  $\mathcal{K}$  if  $y, z \in \mathcal{K}$ ,  $x = y + z \Rightarrow y, z \in \{\lambda x : \lambda \geq 0\}$ .

In what follows, we denote by  $\text{Ext}(\mathcal{K})$ , the set of all nonzero  $x$  for which the corresponding ray is an extreme ray of  $\mathcal{K}$ .

**Proposition 7.** *Let  $\mathcal{C}$  be a closed cone. Then  $\text{Ext}(\mathcal{K}) = \{uu^T : 0 \neq u \in \mathcal{C}\}$ .*

**Proof.** By the description of  $\mathcal{K}$ , it follows that  $\text{Ext}(\mathcal{K}) \subseteq \{uu^T : 0 \neq u \in \mathcal{C}\}$ . Now, suppose that  $0 \neq u \in \mathcal{C}$  and  $uu^T = \sum_1^N u_i u_i^T$ , where  $0 \neq u_i \in \mathcal{C}$ . Then for any  $v \in R^n$  with  $u^T v = 0$ , we have  $\sum_1^N v^T u_i u_i^T v = v^T (uu^T) v = 0$ . This leads to  $u_i^T v = 0$  for all  $i$ . Hence, we have the implication  $u^T v = 0 \Rightarrow u_i^T v = 0$  for every  $i$ . This shows that for any  $i$ ,  $u_i$  is a multiple of  $u$ , that is  $u_i u_i^T$  is a nonnegative multiple of  $uu^T$ . Thus  $uu^T$  is in  $\text{Ext}(\mathcal{K})$ .  $\square$

**Proposition 8.** *Let  $A$  be in  $S^n$  such that every  $2 \times 2$  minor  $A$  is zero. Then, rank of  $A$  is zero or one.*

This is well known and easy to show: Under the given condition on  $A$ , every minor of order  $3 \times 3$ , and by induction, every minor of order  $k \times k$  ( $3 \leq k \leq n$ ) is zero. Hence, rank of  $A$  is zero or one.

### 3. Proofs of Theorems 1 and 2

In this section, we present the proofs of Theorems 1 and 2 and provide some examples. First, we present a preliminary result.

**Proposition 9.** *Let  $\mathcal{C}$  be any closed cone in  $\mathbb{R}^n$ . Then the following statements hold:*

- (a) *For each  $Q \in \text{Aut}(\tilde{\mathcal{C}})$ , we have  $\widehat{Q} \in \text{Aut}(\mathcal{K})$ .*
- (b) *The mapping  $Q \mapsto \widehat{Q} : \text{Aut}(\tilde{\mathcal{C}}) \rightarrow \text{Aut}(\mathcal{K})$  is a group homomorphism.*
- (c) *The mapping  $A \mapsto L_A : \text{Lie}(\text{Aut}(\tilde{\mathcal{C}})) \rightarrow \text{Lie}(\text{Aut}(\mathcal{K}))$  is an injective Lie algebra homomorphism.*

**Proof**

- (a) Let  $Q \in \text{Aut}(\tilde{\mathcal{C}})$ . For any  $u \in \tilde{\mathcal{C}}$ , we have  $Qu \in \tilde{\mathcal{C}}$  and so

$$\widehat{Q}(uu^T) = (Qu)(Qu)^T \in \mathcal{K}.$$

This implies that  $\widehat{Q}(\mathcal{K}) \subseteq \mathcal{K}$ . Since  $Q^{-1} \in \text{Aut}(\tilde{\mathcal{C}})$  and  $\widehat{Q^{-1}} = \widehat{Q}^{-1}$ , we also have  $\widehat{Q}^{-1}(\mathcal{K}) \subseteq \mathcal{K}$ . Thus  $\widehat{Q} \in \text{Aut}(\mathcal{K})$ .

- (b) It is easy to see that the mapping  $Q \mapsto \widehat{Q} : \text{Aut}(\tilde{\mathcal{C}}) \rightarrow \text{Aut}(\mathcal{K})$  is a group homomorphism under multiplication/composition.
- (c) Now, let  $A \in \text{Lie}(\text{Aut}(\tilde{\mathcal{C}}))$  so that there is a differentiable curve  $Q(t)$  in  $\text{Aut}(\tilde{\mathcal{C}})$  with  $Q(0) = I$  (Identity matrix) and  $Q'(0) = A$ . Then by (b),  $L(t) := \widehat{Q(t)}$  is a differentiable curve in  $\text{Aut}(\mathcal{K})$  with  $L(0) = Id$  (Identity transformation). As

$$L'(t)(X) = Q'(t)XQ(t)^T + Q(t)XQ'(t)^T,$$

we see, by putting  $t = 0$ , that  $L'(0) = L_A$ . Thus,  $L_A \in \text{Lie}(\text{Aut}(\mathcal{K}))$ . That the linear mapping  $A \mapsto L_A$  is a Lie algebra homomorphism follows from  $L_{[A,B]} = L_{AB-BA} = L_A L_B - L_B L_A = [L_A, L_B]$ . Finally, the injectivity comes from Proposition 4. This completes the proof.  $\square$

**Proof of Theorem 1.** (a) In view of the above proposition, the mapping  $Q \mapsto \widehat{Q} : \text{Aut}(\tilde{\mathcal{C}}) \rightarrow \text{Aut}(\mathcal{K})$  is a homomorphism. We now show that it is surjective. Let  $L \in \text{Aut}(\mathcal{K})$ . By the Riesz Representation Theorem, there exist matrices  $A_{ij} \in \mathcal{S}^n$  such that for any  $X \in \mathcal{S}^n$ ,

$$L(X) = [A_{ij}, X].$$

Since  $L$  is an automorphism of  $\mathcal{K}$ , it preserves the extreme rays of  $\mathcal{K}$ : For any nonzero  $u$  in  $\mathcal{C}$ , there is a nonzero  $v \in \mathcal{C}$  such that  $L(uu^T) = vv^T$ . Thus,  $u^T A_{ij} u = v_i v_j$  for all  $i, j$  and so

$$(u^T A_{ij} u)(u^T A_{kl} u) = (u^T A_{il} u)(u^T A_{kj} u), \tag{3}$$

for all indices  $i, j, k, l$ , at least three of which are distinct.

Now, fix  $0 \neq x \in \mathbb{R}^n$  and let  $d \in \text{int}(\mathcal{C})$  (which is nonempty by assumption). Then for all small positive  $\varepsilon$ ,  $d + \varepsilon x \in \text{int}(\mathcal{C})$ , hence (3) holds with  $u$  replaced by  $d + \varepsilon x$ . Expanding and comparing terms containing  $\varepsilon^4$ , we get

$$(x^T A_{ij} x)(x^T A_{kl} x) = (x^T A_{il} x)(x^T A_{kj} x).$$

This means that  $L(xx^T) = [x^T A_{ij} x]$  is a matrix with vanishing  $2 \times 2$  minors. By Proposition 8,  $L(xx^T)$  has rank less than or equal to one. As this holds for any nonzero  $x$  in  $\mathbb{R}^n$ , matrices with rank less than or equal to one in  $\mathcal{S}^n$  are mapped, under  $L$ , to matrices of the same type. By a result of Lim [9] or Waterhouse [12], there exists an invertible matrix  $Q \in \mathbb{R}^{n \times n}$  and a real number  $\mu$  such that  $L(X) = \mu XQX^T$  for all  $X \in \mathcal{S}^n$ . Since  $L(uu^T) \in \mathcal{S}_+^n$  for any nonzero  $u \in \mathcal{C}$ ,  $\mu$  cannot be negative. Also,  $\mu$  cannot be zero, as  $L$  is invertible. We may assume that  $\mu = 1$ . Thus, there exists an invertible matrix  $Q$  such that

$$L(X) = QXQ^T \quad (X \in S^n).$$

Now, let  $0 \neq u \in \mathcal{C}$ . As  $L$  preserves  $\text{Ext}(\mathcal{K})$ ,  $(Qu)(Qu)^T = L(uu^T) = vv^T$  for some  $0 \neq v \in \mathcal{C}$ . From Proposition 6,  $Qu = v \in \mathcal{C}$  or  $Qu = -v \in -\mathcal{C}$ . This shows that  $Q(\mathcal{C}) \subseteq \mathcal{C} \cup -\mathcal{C}$  and hence

$$Q(\tilde{\mathcal{C}}) \subseteq \tilde{\mathcal{C}}.$$

Now, applying this argument to  $L^{-1}$  and to the corresponding  $Q^{-1}$ , we get  $Q^{-1}(\tilde{\mathcal{C}}) \subseteq \tilde{\mathcal{C}}$ . This means that  $Q \in \text{Aut}(\tilde{\mathcal{C}})$ . Thus, we have shown that

$$L = \widehat{Q}, \quad \text{where } Q \in \text{Aut}(\tilde{\mathcal{C}}).$$

It is clear that for any  $Q \in \text{Aut}(\tilde{\mathcal{C}})$ ,  $-Q \in \text{Aut}(\tilde{\mathcal{C}})$  and  $\widehat{Q} = \widehat{-Q}$ . In view of Proposition 4, each element of  $\text{Aut}(\mathcal{K})$  has exactly two pre-images in  $\text{Aut}(\tilde{\mathcal{C}})$ . This completes the proof of (a).

(b) In view of the previous proposition, we need only to show that the specified mapping is surjective. Let  $L \in \text{Lie}(\text{Aut}(\mathcal{K}))$ . By Lemma 10 given below, there is a differentiable curve  $Q(t)$  in  $\text{Aut}(\tilde{\mathcal{C}})$  such that  $Q(0) = I$  and  $e^{tL}(X) = Q(t)XQ(t)^T$  for all  $X \in S^n$ . Now, differentiating both sides of  $e^{tL}(X) = Q(t)XQ(t)^T$  (for any fixed  $X$ ) and evaluating the derivatives at  $t = 0$ , we get

$$L(X) = AX + XA^T \quad (X \in S^n),$$

where  $A = Q'(0)$ . By definition,  $A \in \text{Lie}(\text{Aut}(\tilde{\mathcal{C}}))$ . Thus,  $L = L_A$  with  $A \in \text{Lie}(\text{Aut}(\tilde{\mathcal{C}}))$ . This completes the proof of (b).  $\square$

**Lemma 10.** *Suppose that the mapping  $Q \mapsto \widehat{Q} : \text{Aut}(\tilde{\mathcal{C}}) \rightarrow \text{Aut}(\mathcal{K})$  is a two-to-one surjective mapping. Let  $\epsilon > 0$  so that the open balls  $B(I, \epsilon)$  and  $B(-I, \epsilon)$  around  $I$  and  $-I$  respectively, are disjoint in  $R^{n \times n}$ . Then for any  $L \in \text{Lie}(\text{Aut}(\mathcal{K}))$ , there is a  $\delta > 0$  and a (unique) differentiable curve  $Q(t) : (-\delta, \delta) \rightarrow \text{Aut}(\tilde{\mathcal{C}}) \cap B(I, \frac{1}{2}\epsilon)$  such that  $Q(0) = I$  and*

$$e^{tL} = \widehat{Q(t)} \quad \forall t \in (-\delta, \delta).$$

Furthermore, if  $\mathcal{C}$  is pointed, we may choose  $\epsilon > 0$  and  $\delta > 0$  so that

$$Q(t) : (-\delta, \delta) \rightarrow \text{Aut}(\mathcal{C}) \cap B\left(I, \frac{1}{2}\epsilon\right).$$

**Proof.** Let  $L \in \text{Lie}(\text{Aut}(\mathcal{K}))$  so that  $e^{tL} \in \text{Aut}(\mathcal{K})$  for all  $t$ . By our assumption, there exists  $Q(t) \in \text{Aut}(\tilde{\mathcal{C}})$  such that  $e^{tL}(X) = Q(t)XQ(t)^T$  for all  $X \in S^n$ . We see that  $e^{tL}(I) = Q(t)Q(t)^T$ ; this shows that  $\{Q(t) : -1 \leq t \leq 1\}$  is a bounded set in  $R^{n \times n}$ . Now choose  $\delta > 0$  so that for  $t \in (-\delta, \delta)$ ,  $Q(t) \in B(I, \epsilon)$  or  $Q(t) \in B(-I, \epsilon)$ . (If this is not true, then by using the boundedness of  $\{Q(t) : -1 \leq t \leq 1\}$  and taking appropriate limits, we get a  $Q$  that is outside these balls satisfying  $Id(X) = QXQ^T$  for all  $X \in S^n$ . This would contradict Proposition 4.)

Note that if  $Q(t) \in B(-I, \epsilon)$ , then  $-Q(t) \in B(I, \epsilon)$ . Since the mapping  $Q \mapsto \widehat{Q}$  is two-to-one mapping, in  $B(I, \epsilon)$  we have exactly one  $Q(t)$  for each  $t$ .

Thus, we may assume that for each  $t \in (-\delta, \delta)$ , there is a unique  $Q(t) \in B(I, \frac{1}{2}\epsilon) \cap \text{Aut}(\tilde{\mathcal{C}})$ . We now claim that this  $Q(t)$  is continuous in  $t$ . Suppose  $t_k \rightarrow \bar{t}$  in  $(-\delta, \delta)$  and (because of boundedness)  $Q(t_{k_m}) \rightarrow \bar{Q} \neq Q(\bar{t})$ . But then

$$\bar{Q}X\bar{Q}^T = \lim e^{t_{k_m}L}(X) = e^{\bar{t}L}(X) = Q(\bar{t})XQ(\bar{t})^T \quad (X \in S^n)$$

implies that  $\bar{Q} = Q(\bar{t})$  by Proposition 4 and uniqueness in  $B(I, \epsilon)$ . This contradiction proves continuity. Now, by taking a smaller  $\delta$  (if necessary), we show that  $Q(t)$  is differentiable on  $(-\delta, \delta)$ . We show this by proving the differentiability of the first column of  $Q(t)$  and repeating the argument for other columns. Let  $e_1$  be the vector in  $R^n$  with one in the first slot and zeros elsewhere and  $E_1 := e_1e_1^T$ . Let  $Q(t)e_1 = v(t)$  so that

$$e^{tL}(E_1) = v(t)v(t)^T.$$

Let  $\alpha(t)$  denote the first component of  $v(t)$ . Now, for all  $t$  near zero, the  $(1, 1)$  component of  $e^{tL}(E_1)$ , namely  $e^{tL}(E_1)_{11}$ , is close to one and differentiable in  $t$ ; thus,  $\alpha(t)^2 = e^{tL}(E_1)_{11}$  is nonzero and differentiable at all points near zero. As  $\alpha(t)$  is continuous, nonzero, and  $\alpha(t)^2$  is differentiable,  $\alpha(t)$  is also differentiable near zero. Now,  $v(t)$  is  $\frac{1}{\alpha(t)}$  times the first column of  $e^{tL}(E_1)$ , hence differentiable at all points near zero. By a similar argument, we see that all columns of  $Q(t)$  are differentiable near zero. Thus,  $Q(t)$  is differentiable on some  $(-\delta, \delta)$ .

Now suppose that  $\mathcal{C}$  is pointed. The stated conclusion about  $Q(t)$  follows once we show that for all small  $\varepsilon > 0$ ,

$$Aut(\tilde{\mathcal{C}}) \cap B(I, \varepsilon) \subseteq Aut(\mathcal{C}).$$

Assuming this inclusion to be false for every  $\varepsilon$ , we can find sequences  $x_k \in \mathcal{C}$ ,  $Q_k \in Aut(\tilde{\mathcal{C}})$  such that  $Q_k \rightarrow I$  and  $Q_k(x_k) \notin \mathcal{C}$ . We may assume that  $\|x_k\| = 1$  for all  $k$  and let  $\lim x_k = x \in \mathcal{C}$ . As  $Q_k(x_k) \in -\mathcal{C}$ , taking limits, we get  $I(x) \in -\mathcal{C}$ . Thus,  $x \in \mathcal{C} \cap -\mathcal{C}$ . Since  $\|x\| = 1$ , we reach a contradiction to the pointedness of  $\mathcal{C}$ . We thus have the inclusion and the proof is complete.  $\square$

**Proof of Theorem 2.** Suppose that  $\mathcal{C}$  is pointed and has nonempty interior. To see Item (i), we proceed as in the proof of Theorem 1. As  $Aut(\mathcal{C})$  is a subgroup of  $Aut(\tilde{\mathcal{C}})$  and  $Lie(Aut(\mathcal{C})) \subseteq Lie(Aut(\tilde{\mathcal{C}}))$ , the mapping  $A \mapsto L_A : Lie(Aut(\mathcal{C})) \rightarrow Lie(Aut(\mathcal{K}))$  is an injective Lie algebra homomorphism. To show that this map is surjective, let  $L \in Lie(Aut(\mathcal{K}))$ . Then we have  $e^{tL} \in Aut(\mathcal{K})$  for all  $t \in \mathbb{R}$ . By Theorem 1 and Lemma 10, there is a differentiable curve  $Q(t)$  in  $Aut(\mathcal{C})$  such that  $Q(0) = I$  and  $e^{tL} = \widehat{Q(t)}$  for all  $t$  near zero. By repeating the proof of part (b) in Theorem 1, we verify that  $L = L_A$ , where,  $A = Q'(0)$  now belongs to  $Lie(Aut(\mathcal{C}))$ . This completes the proof of (i).

Now suppose, additionally, that  $\mathcal{C} \setminus \{0\}$  is connected. Since  $Aut(\mathcal{C})$  is a subgroup of  $Aut(\tilde{\mathcal{C}})$ , the mapping  $Q \mapsto \widehat{Q} : Aut(\mathcal{C}) \rightarrow Aut(\mathcal{K})$  is a homomorphism. We now show that this map is surjective and injective. Let  $L \in Aut(\mathcal{K})$ . Since  $int(\mathcal{C}) \neq \emptyset$  we can apply Theorem 1 and get a  $Q \in Aut(\tilde{\mathcal{C}})$  such that  $L = \widehat{Q}$ . Then  $Q(\tilde{\mathcal{C}}) = \tilde{\mathcal{C}}$  implies that  $Q(\mathcal{C}) \subseteq \mathcal{C} \cup -\mathcal{C}$  and by the invertibility of  $Q$ ,

$$Q(\mathcal{C} \setminus \{0\}) \subseteq (\mathcal{C} \setminus \{0\}) \cup -(\mathcal{C} \setminus \{0\}).$$

Since  $\mathcal{C}$  is pointed, the sets  $\mathcal{C} \setminus \{0\}$  and  $-(\mathcal{C} \setminus \{0\})$  are separated (in the sense that each is disjoint from the closure of the other). By our assumption,  $\mathcal{C} \setminus \{0\}$  is connected; hence  $Q(\mathcal{C} \setminus \{0\})$  is also connected. It follows that  $Q(\mathcal{C} \setminus \{0\}) \subseteq (\mathcal{C} \setminus \{0\})$  or  $Q(\mathcal{C} \setminus \{0\}) \subseteq -(\mathcal{C} \setminus \{0\})$  and by taking closures,  $Q(\mathcal{C}) \subseteq \mathcal{C}$  or  $Q(\mathcal{C}) \subseteq -\mathcal{C}$ ; As  $Q$  and  $-Q$  define the same  $L$ , without loss of generality, we may assume that  $Q(\mathcal{C}) \subseteq \mathcal{C}$ . Now, working with  $L^{-1}$  and  $Q^{-1}$ , we get  $Q^{-1}(\mathcal{C}) \subseteq \mathcal{C}$  or  $Q^{-1}(\mathcal{C}) \subseteq -\mathcal{C}$ . Since  $\mathcal{C}$  is pointed and has nonzero elements, we cannot have  $Q(\mathcal{C}) \subseteq \mathcal{C}$  and  $Q^{-1}(\mathcal{C}) \subseteq -\mathcal{C}$ . Thus,  $Q^{-1}(\mathcal{C}) \subseteq \mathcal{C}$ . Hence,  $Q(\mathcal{C}) = \mathcal{C}$ , that is,  $Q \in Aut(\mathcal{C})$ . Thus, we have shown that for each  $L \in Aut(\mathcal{K})$ , there is a  $Q \in Aut(\mathcal{C})$  such that  $L = \widehat{Q}$ .

Now for the uniqueness: Now let  $P \in Aut(\mathcal{C})$  such that  $L = \widehat{P}$ . Then, by Proposition 4,  $P = \pm Q$ . If  $P = -Q$ , then we have the equality  $-\mathcal{C} = -Q(\mathcal{C}) = P(\mathcal{C}) = \mathcal{C}$ . However, this cannot happen since  $\mathcal{C}$  is pointed and has nonzero elements. Hence we must have  $P = Q$ . This establishes (ii).

Finally, let  $\mathcal{C}$  be a proper cone. Then  $\mathcal{C}$  is pointed and  $\mathcal{C} \setminus \{0\}$  is convex. Also,  $int(\mathcal{C}) \neq \emptyset$ . Thus, all the conditions of Theorem 2 are satisfied. Hence we have statements (i) and (ii). This completes the proof.  $\square$

**Remark.** The proof of Theorem 2 given above actually reveals the following: Suppose  $\mathcal{C}$  is a closed pointed cone with nonempty interior and  $\mathcal{C} \setminus \{0\}$  is connected. Then

$$Aut(\tilde{\mathcal{C}}) = Aut(\mathcal{C}) \cup -Aut(\mathcal{C}).$$

We now provide some examples to illustrate our results.

**Example 1.** Let  $\mathcal{C} = R^n$  (or the closed upper half-space in  $R^n$ ). Then Theorem 1 is applicable and we get the results mentioned in the Introduction.  $\square$

We note that Theorem 2 is applicable to any self-dual cone  $\mathcal{C}$  (that is,  $\mathcal{C} = \mathcal{C}^*$ ), in particular, to symmetric cones in Euclidean algebras [5].

**Example 2.** Let  $\mathcal{C} = R_+^n$ . In this case,  $\mathcal{K}$  is the set of all completely positive matrices [3] and  $\mathcal{E}$  is the set of all symmetric copositive matrices. It is well known that every automorphism of  $R_+^n$  is a product of a permutation matrix and a diagonal matrix with positive diagonal. In addition, it easily follows from Theorem 3 that  $Lie(Aut(R_+^n))$  is the set of all  $n \times n$  diagonal matrices. Now, applying Theorem 2, one can describe  $Aut(\mathcal{K})$  and its Lie algebra.  $\square$

The following examples show that the mappings  $Q \mapsto \widehat{Q}$  and  $A \mapsto L_A$  in Theorems 1 and 2 need not be surjective without appropriate conditions on  $\mathcal{C}$ .

**Example 3.** For  $n > 2$ , let  $\mathcal{L}_+^n$  denote the so-called ice-cream cone (or the second order cone) given by

$$\mathcal{L}_+^n = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in R^n : t \in R, x \in R^{n-1}, t \geq \|x\| \right\}$$

and let

$$\mathcal{C} := \partial \mathcal{L}_+^n = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in R^n : t = \|x\| \right\}.$$

Clearly  $\mathcal{C}$  is pointed and  $\mathcal{C} \setminus \{0\}$  is connected, but  $int(\mathcal{C}) = \emptyset$ . (It may be instructive to visualize  $\mathcal{C}$  in  $R^3$ .) Let  $J_n = diag(1, -1, \dots, -1) \in R^{n \times n}$  and  $\Gamma(X) := \langle X, J_n \rangle$  for any  $X \in S^n$ . Since  $\mathcal{K} - \mathcal{K} \subseteq \ker(\Gamma)$ , we see that  $\mathcal{K} - \mathcal{K} \neq S^n$ .

Let  $L$  be an invertible linear transformation on  $S^n$  such that  $L$  coincides with the Identity transformation on  $\mathcal{K} - \mathcal{K}$ , but not on the entire  $S^n$ . (For example, writing  $S^n = (\mathcal{K} - \mathcal{K}) \oplus (\mathcal{K} - \mathcal{K})^\perp$ , we may define  $L(x + y) = x + 2y$  for  $x \in \mathcal{K} - \mathcal{K}$  and  $y \in (\mathcal{K} - \mathcal{K})^\perp$ .) Then  $L \in Aut(\mathcal{K})$ . Assume that there is a  $Q \in Aut(\widehat{\mathcal{C}})$  such that  $L = \widehat{Q}$ , that is,  $L(X) = QXQ^T$  for every  $X \in S^n$ . Then for all  $u \in \mathcal{C}$ ,  $uu^T = L(uu^T) = Quu^TQ^T = (Qu)(Qu)^T$ . By Proposition 6,  $Q(u) = \pm u$ . Thus,  $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ , where  $\mathcal{C}_1 := \{u \in \mathcal{C} : Qu = u\}$  and  $\mathcal{C}_2 := \{u \in \mathcal{C} : Qu = -u\}$ . Since  $\mathcal{C}_1 \setminus \{0\}$  and  $\mathcal{C}_2 \setminus \{0\}$  are separated, and

$$\mathcal{C} \setminus \{0\} = \mathcal{C}_1 \setminus \{0\} \cup \mathcal{C}_2 \setminus \{0\},$$

by connectedness of  $\mathcal{C} \setminus \{0\}$ , we get  $\mathcal{C} \subseteq \mathcal{C}_1$  or  $\mathcal{C} \subseteq \mathcal{C}_2$ , i.e.,  $Q = \pm I$  on  $\mathcal{C}$ . If  $Q = I$  on  $\mathcal{C}$ , by linearity,  $Q = I$  on  $\mathcal{L}_+^n$ . As  $\mathcal{L}_+^n - \mathcal{L}_+^n = R^n$ ,  $Q = I$  on  $R^n$ . This yields  $L = \widehat{Q} = \widehat{I} = Id$  which is a contradiction. Similarly, if  $Q = -I$  on  $\mathcal{C}$ , we get  $Q = -I$  on  $R^n$  and hence  $L = Id$ , which, once again, is a contradiction. This shows that the mapping  $Q \mapsto \widehat{Q} : Aut(\widehat{\mathcal{C}}) \rightarrow Aut(\mathcal{K})$  is not surjective. Thus, statements (a) in Theorem 1 and (ii) in Theorem 2 fail to hold.

Now, for  $t \in (-1, 1)$ , consider the differentiable curve  $L(t) : (\mathcal{K} - \mathcal{K}) \oplus (\mathcal{K} - \mathcal{K})^\perp \rightarrow (\mathcal{K} - \mathcal{K}) \oplus (\mathcal{K} - \mathcal{K})^\perp$  given by

$$L(t)(x + y) := x + (1 + t)y \quad (x \in \mathcal{K} - \mathcal{K}, y \in (\mathcal{K} - \mathcal{K})^\perp).$$

Then  $L(t) \in Aut(\mathcal{K})$  and  $L(0) = Id$ . By definition,  $L_1 := L'(0) \in Lie(Aut(\mathcal{K}))$ . Note that

$$L_1(x) = 0 \text{ for all } x \in \mathcal{K} - \mathcal{K} \text{ and } L_1(y) = y \text{ for all } y \in (\mathcal{K} - \mathcal{K})^\perp.$$

Suppose, if possible,  $L_1 = L_A$  for some  $A \in R^{n \times n}$ . Since  $L_1(\mathcal{K} - \mathcal{K}) = \{0\}$ , for any  $u \in \mathcal{C}$ ,

$$0 = L_1(uu^T) = Auu^T + uu^TA^T.$$

It follows that for any  $x \in int(\mathcal{L}_+^n)$ ,

$$x^T(Auu^T + uu^TA^T)x = 0.$$

Thus,  $u^T x (x^T A u) = 0$ . As  $u^T x > 0$  for any  $x \in \text{int}(\mathcal{L}_+^n)$  and  $0 \neq u \in \mathcal{C}$ , we must have  $x^T A u = 0$  for all such  $x$  and  $u$ . Since  $\mathcal{L}_+^n - \mathcal{L}_+^n = R^n$ ,  $x^T A u = 0$  for all  $x \in R^n$  and  $u \in \mathcal{C}$ ; thus, for any  $u \in \mathcal{C}$ ,  $Au = 0$ . Again, by linearity,  $Au = 0$  for all  $u \in \mathcal{L}_+^n$  and  $Au = 0$  for any  $u \in R^n$ . Thus,  $A = 0$ . This gives  $L_1 = L_A = 0$ . This is not possible, as  $L_1(y) = y$  for all  $y \in (\mathcal{K} - \mathcal{K})^\perp$ . Hence, statements (b) in Theorem 1 and (i) in Theorem 2 fail to hold.  $\square$

**Example 4.** Let  $\mathcal{C}$  be the closed upper half-plane in  $R^2$ . Then  $\mathcal{C}$  has nonempty interior,  $\mathcal{C} \setminus \{0\}$  is connected, but  $\mathcal{C}$  is not pointed. We show that the mappings in Items (i) and (ii) of Theorem 2 are not surjective.

It is clear that every (symmetric)  $2 \times 2$  matrix that is copositive on  $\mathcal{C}$  is also positive semidefinite; hence  $\mathcal{E} = S_+^2$  and so  $\mathcal{K} = S_+^2$ . By the result mentioned in the Introduction,

$$\text{Aut}(\mathcal{K}) = \{ \widehat{Q} : Q \text{ invertible in } R^{2 \times 2} \}$$

and

$$\text{Lie}(\text{Aut}(\mathcal{K})) = \{ L_A : A \in R^{2 \times 2} \}.$$

For the cone  $\mathcal{C}$ , it is easily verified that

$$\text{Aut}(\mathcal{C}) = \left\{ A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : c > 0, a \neq 0 \right\}$$

and

$$\text{Lie}(\text{Aut}(\mathcal{C})) = \left\{ B = \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} : p, q, r \in R \right\}.$$

Now let

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

For this  $Q$ , it is easily verified (using Proposition 4) that  $\widehat{Q}$ , which is in  $\text{Aut}(\mathcal{K})$ , is not of the form  $\widehat{A}$  for any  $A \in \text{Aut}(\mathcal{C})$ . Also,  $L_Q$ , which belongs to  $\text{Lie}(\text{Aut}(\mathcal{K}))$ , is not of the form  $L_B$  for any  $B \in \text{Lie}(\text{Aut}(\mathcal{C}))$ .  $\square$

**Example 5.** Let  $\mathcal{C} = R_+^2 \cup \{ \lambda f : \lambda \geq 0 \}$ , where  $f = [-1 \ 1]^T$ . Then  $\mathcal{C}$  is pointed, has nonempty interior, but  $\mathcal{C} \setminus \{0\}$  is not connected.

Now it is easy to see that  $\text{Aut}(\mathcal{C}) \subseteq \text{Aut}(R_+^2)$  and based on the description of elements in  $\text{Aut}(R_+^2)$  (see Example 2),

$$\text{Aut}(\mathcal{C}) = \left\{ A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha > 0 \right\}.$$

Let

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then  $Q(f) = -f$  and  $Q \in \text{Aut}(R_+^2)$  and so,  $\widehat{Q}(\mathcal{K}) \subseteq \mathcal{K}$ . As  $Q^{-1} = Q$ , we must have  $\widehat{Q} \in \text{Aut}(\mathcal{K})$ . By Proposition 4,  $\widehat{Q}$  is not of the form  $\widehat{A}$  for any  $A \in \text{Aut}(\mathcal{C})$ . Thus, the mapping in Item (ii) of Theorem 2 is not surjective.  $\square$

#### 4. The copositive cone

Recall that  $\mathcal{E}$  denotes the copositive cone of  $\mathcal{C}$ . In this section we describe the elements of  $\text{Aut}(\mathcal{E})$  and its Lie algebra.

It is easily seen that for any closed convex cone  $K$  in (the real Hilbert space)  $H$ ,

$$L \in \text{Aut}(K) \Leftrightarrow L^* \in \text{Aut}(K^*),$$

where  $L^*$  denotes the adjoint/transpose of  $L$ . This equivalence, along with the equality  $(e^{tL})^* = e^{tL^*}$  for any  $t \in \mathbb{R}$  shows that

$$L \in \text{Lie}(\text{Aut}(K)) \Leftrightarrow L^* \in \text{Lie}(\text{Aut}(K^*)).$$

When specialized to a completely positive cone, we get the following.

**Proposition 11.** *Let  $\mathcal{C}$  be any closed cone in  $\mathbb{R}^n$ . Then*

- (i)  $L \in \text{Aut}(\mathcal{C}) \Leftrightarrow L^* \in \text{Aut}(\mathcal{E})$ .
- (ii)  $L \in \text{Lie}(\text{Aut}(\mathcal{C})) \Leftrightarrow L^* \in \text{Lie}(\text{Aut}(\mathcal{E}))$ .

This proposition, coupled with Theorems 1 and 2, will allow us to describe the automorphisms of  $\mathcal{E}$  and the corresponding Lie algebra. Here is a sample result.

**Corollary 12.** *Suppose  $\mathcal{C}$  is a closed pointed cone with nonempty interior and  $\mathcal{C} \setminus \{0\}$  is connected. Then every  $L \in \text{Aut}(\mathcal{E})$  is given by*

$$L(X) = Q^T X Q \quad (X \in \mathcal{S}^n)$$

for some  $Q \in \text{Aut}(\mathcal{C})$  and every  $L \in \text{Lie}(\text{Aut}(\mathcal{E}))$  is of the form  $L = L_{AT}$  for some  $A \in \text{Lie}(\text{Aut}(\mathcal{C}))$ .

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