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A Generalization of the Nash Equilibrium Theorem on Bimatrix Games

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Abstract: In this article, we consider a two-person game in which the first player picks a row representative matrix M from a nonempty set $\mathcal{A}$ of $m \times n$ matrices and a probability distribution x on $\{1, 2, \ldots, m\}$ while the second player picks a column representative matrix N from a nonempty set $\mathcal{B}$ of $m \times n$ matrices and a probability distribution y on $\{1, 2, \ldots, n\}$. This leads to the respective costs of $x^T My$ and $x^T Ny$ for these players. We establish the existence of an ε-equilibrium for this game under the assumption that $\mathcal{A}$ and $\mathcal{B}$ are bounded. When the sets $\mathcal{A}$ and $\mathcal{B}$ are compact in $\mathbb{R}^{m \times n}$, the result yields an equilibrium state at which stage no player can decrease his cost by unilaterally changing his row/column selection and probability distribution. The result, when further specialized to singleton sets, reduces to the famous theorem of Nash on bimatrix games.

Key Words: Bimatrix game, ε-equilibrium, optimal strategies, vertical linear complementarity problem, degree, stability

1 Introduction

The celebrated result of Nash (1951) on the existence of an equilibrium pair in mixed strategies for a bimatrix game states that given two real $m \times n$ matrices A and B, there exists a pair – called the equilibrium pair – of mixed strategies (i.e. probability vectors) $\bar{x}$ and $\bar{y}$ such that

$$\bar{x}^T A\bar{y} \leq u^T A\bar{y} \quad \text{and} \quad \bar{x}^T B\bar{y} \leq \bar{x}^T Bv \tag{1}$$

for all mixed strategies $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Nash proved this result via the Brouwer Fixed Point Theorem. Lemke and Howson (1964) gave a constructive proof of the existence of equilibrium pairs which led to the development of the famous Lemke's algorithm for solving linear complementarity problems, see Lemke (1965), Cottle, Pang and Stone (1992).

In this article, we prove an extension of the above theorem of Nash. To explain this extension, we consider two nonempty sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{R}^{m \times n}$. An $m \times n$ matrix R is said to be a *row representative* of $\mathcal{A}$ if for each $i = 1, 2, \ldots, m$, $R_i \in \{A_i : A \in \mathcal{A}\}$ where the subscript denotes the corresponding row. Similarly, an $m \times n$ matrix S is a *column representative* of $\mathcal{B}$ if for each $j = 1, 2, \ldots, n$, $S^j \in \{B^j : B \in \mathcal{B}\}$ where the superscript denotes the corresponding column. Our result is

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Theorem 1: Suppose that the nonempty sets $\mathscr{A}$ and $\mathscr{B}$ are bounded in $\mathbb{R}^{m\times n}$. Then there exists a pair $(\bar{x}, \bar{y})$ of probability vectors with the following property: for every $\varepsilon > 0$ there exist a row representative M_ε of $\mathscr{A}$, and a column representative N_ε of $\mathscr{B}$ such that

$$\bar{x}^T M_\varepsilon \bar{y} \leq (1+\varepsilon) u^T R \bar{y} \quad \text{and} \quad \bar{x}^T N_\varepsilon \bar{y} \leq (1+\varepsilon) \bar{x}^T S v \tag{2}$$

or alternatively,

$$\bar{x}^T M_\varepsilon \bar{y} \leq u^T R \bar{y} + \varepsilon \quad \text{and} \quad \bar{x}^T N_\varepsilon \bar{y} \leq \bar{x}^T S v + \varepsilon \tag{3}$$

for all probability vectors u and v, for all row representative matrices R of $\mathscr{A}$, and for all column representative matrices S of $\mathscr{B}$.

Moreover, when $\mathscr{A}$ and $\mathscr{B}$ are compact, the above statements hold with $\varepsilon = 0$.

To see the game theoretic implication of the above result, consider two players, I and II. Player I deals with the rows of matrices in $\mathscr{A}$ while player II deals with the columns of matrices in $\mathscr{B}$. In the game denoted by $\Gamma(\mathscr{A}, \mathscr{B})$, player I chooses a row representative M of $\mathscr{A}$ and a probability distribution x on the set $\{1, 2, \ldots, m\}$. Player II chooses a column representative N of $\mathscr{B}$ and a probability distribution y on the set $\{1, 2, \ldots, n\}$. Then the cost for the first player is $x^T M y$ while that for the second player is $x^T N y$. The above theorem describes for $\Gamma(\mathscr{A}, \mathscr{B})$ the existence of an *ε-equilibrium* at which stage no player can decrease his cost (modulo ε) by unilaterally changing his row/column selection and probability distribution.

Our proof of Theorem 1 is based on degree theory. The main idea is to show that a certain complementarity system (equation) has a solution by explicitly computing the degree of the defining function. While the degree theory is a standard tool in nonlinear analysis, apparently it has not been used in the analysis of conflict situations. In Cottle, Pang, and Stone (1992), Chapter 6, a geometric and degree theoretic proof of Nash's theorem on bimatrix games is given; it is not at all clear whether this proof can be extended to cover our present situation. The analytical and degree theoretic analysis presented in this paper has the additional advantage (apart from proving the existence of an equilibrium pair) in that it allows us, as we illustrate in Section 5, to study stability aspects of the game under consideration.

2 Preliminaries

Throughout this paper, '$\wedge$' refers to the componentwise minimum, e denotes the vector of ones in $\mathbb{R}^m$ or $\mathbb{R}^n$, and 'complementarity' refers to a condition in which a minimum of certain vectors is set to zero. We shall say that a matrix (or a vector) is positive if all its entries are positive. Norm of a vector always refers to the

l_1-norm. We do not specify any particular norm for a matrix but simply note that when a set in $\mathbb{R}^{m \times n}$ is bounded, the set of all entries of matrices in this set are uniformly bounded below and above. A vector is a probability vector if all its entries are nonnegative and its l_1-norm is one.

We briefly describe the degree theoretic properties that are needed in this paper. The standard references for degree theory are Lloyd (1978) and Ortega and Rheinboldt (1970).

Following Ortega and Rheinboldt (1970), Chapter 6, corresponding to a bounded open set Ω in $\mathbb{R}^k$ and a continuous function $f: \bar{\Omega} \to \mathbb{R}^k$ with $0 \notin f(\partial\Omega)$, we denote the *degree of f at* 0 *relative to* Ω by $\deg(f, \Omega, 0)$. We list below properties that are relevant to our discussion.

1. If $\deg(f, \Omega, 0) \neq 0$, then the equation $f(z) = 0$ has a solution in Ω.
2. (Nearness property) Suppose that $\deg(f, \Omega, 0)$ is defined. If g is a continuous function on $\bar{\Omega}$ such that

$$\sup_{x \in \Omega} \| g(x) - f(x) \| < \operatorname{dist}(0, f(\partial\Omega)) \tag{4}$$

then $\deg(g, \Omega, 0)$ is defined and is equal to $\deg(f, \Omega, 0)$, where the right hand side of (4) denotes the distance between zero and compact set $f(\partial\Omega)$.

3. (Excision-addition property) Suppose that $\deg(f, \Omega, 0)$ is defined.
Let $U_1, U_2, \ldots, U_l$ be disjoint bounded open sets contained in Ω such that $f(z) = 0$ has no solution in $\bar{\Omega} \backslash (\overline{\cup_1^l U_i})$. Then

$$\deg(f, \Omega, 0) = \sum_{i=1}^{l} \deg(f, U_i, 0). \tag{5}$$

4. (Homotopy invariance property) Suppose that $H: [0, 1] \times \bar{\Omega} \to \mathbb{R}^k$ is continuous and $0 \notin H(t, \partial\Omega)$ for all $t \in [0, 1]$. Then

$$\deg(H(0, \cdot), \Omega, 0) = \deg(H(1, \cdot), \Omega, 0).$$

For ease of notation, we shall write $H_t(\cdot)$ instead of $H(t, \cdot)$.

5. Suppose that $f(z) = 0$ has a unique solution z^* in Ω, and at z^*, f is differentiable with a nonsingular Jacobian matrix $f'(z^*)$. Then

$$\deg(f, \Omega, 0) = \operatorname{sgn} \det f'(z^*). \tag{6}$$

Given a $k \times k$ matrix Q and a vector $r \in \mathbb{R}^k$, let $G(z) := z \wedge (Qz + r)$. The equation $G(z) = 0$ describes the linear complementarity problem LCP(Q, r): find a vector $z \in \mathbb{R}^k$ such that

$$z \geq 0, \quad Qz + r \geq 0 \quad \text{and} \quad z^T(Qz + r) = 0. \tag{7}$$

Suppose that $\hat{z}$ is the unique solution of $G(z)=0$ in a bounded open set $\Omega \subset \mathbb{R}^k$ and $\hat{z}+Q\hat{z}+r>0$. (In the LCP terminology, $\hat{z}$ is a unique and nondegenerate solution of LCP(Q,r).) If $I := \{i : \hat{z}_i \neq 0\}$ and Q_{II} denotes the submatrix of Q corresponding to the indexes in I, then Q_{II} is nonsingular (Mangasarian (1980), Cor. 3.2) and it is easily seen that G is differentiable at $\hat{z}$ with $\det G'(\hat{z}) = \det Q_{II}$. In this situation,

$$\deg(G,\Omega,0) = \operatorname{sgn}\det Q_{II}. \tag{8}$$

We refer the reader to Gowda (1993) for some recent applications of degree theory to linear complementarity problems.

3 A Complementarity System

From now onwards, we assume that $\mathscr{A}$ and $\mathscr{B}$ are two nonempty bounded subsets of $\mathbb{R}^{m\times n}$. Then for arbitrary positive vectors $p\in\mathbb{R}^m$ and $q\in\mathbb{R}^n$, the functions

$$\phi_p(y) := \inf_{A\in\mathscr{A}}(Ay-p) \quad \text{and} \quad \psi_q(x) := \inf_{B\in\mathscr{B}}(B^Tx-q)$$

are well defined, where the 'inf' refers to the componentwise infimum. Central to our analysis is the function $F_{(p,q)}$ defined by

$$F_{(p,q)}(z) = \begin{bmatrix} x\wedge\phi_p(y) \\ y\wedge\psi_q(x) \end{bmatrix} \quad \text{where} \quad z = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{9}$$

For ease of notation, we shall write $z=(x,y)$ and $F := F_{(e,e)}$.

In the result below, we tie the existence of an ε-equilibrium pair as stipulated in Theorem 1 with the existence of a zero of F.

Proposition 1: Suppose that the entries of matrices of $\mathscr{A}$ *and* $\mathscr{B}$ are uniformly bounded with a positive lower bound. Then there exists a pair of probability vectors $(\bar{x},\bar{y})$ satisfying the conclusion of Theorem 1 if and only if the equation $F(z)=0$ has a solution.

Proof: Let (x,y) be a solution of $F(z)=0$, so that

$$x\wedge \inf_{A\in\mathscr{A}}(Ay-e)=0 \quad \text{and} \quad y\wedge \inf_{B\in\mathscr{B}}(B^Tx-e)=0. \tag{10}$$

Clearly, x and y are nonnegative and nonzero. Let $\varepsilon>0$ be given. We construct a row representative matrix M_ε. If $x_i=0$, let $(M_\varepsilon)_i$ be any A_i with $A\in\mathscr{A}$. If $x_i>0$, then let $(M_\varepsilon)_i\in\{A_i : A\in\mathscr{A}\}$ be such that $0\le(M_\varepsilon y)_i-1\le\varepsilon$; such an $(M_\varepsilon)_i$ exists since $\inf_{\mathscr{A}}[Ay-e]_i=0$. Then

$$x^T(M_\varepsilon y-e)=\sum_{i=1}^{m}x_i(M_\varepsilon y-e)_i=\sum_{x_j>0}x_j((M_\varepsilon y)_j-1)\leq\varepsilon\sum_{x_j>0}x_j=\varepsilon\|x\|.$$

Thus $x^TM_\varepsilon y\leq(1+\varepsilon)\|x\|$. Since $x\neq 0$, upon putting $\bar{x}=x/\|x\|$, we get $\bar{x}^TM_\varepsilon y\leq(1+\varepsilon)$ where $\bar{x}$ is a probability vector. Similarly, we get the existence of a column representative N_ε of $\mathscr{B}$ satisfying $\bar{y}^TN_\varepsilon^Tx\leq(1+\varepsilon)$ with $\bar{y}=y/\|y\|$. From (10), it follows that for every $A\in\mathscr{A}$, $(Ay-e)\geq 0$, whence $Ry\geq e$ for any row representative matrix R of $\mathscr{A}$. We thus have $(\bar{x}^TM_\varepsilon y)e\leq(1+\varepsilon)e\leq(1+\varepsilon)Ry$ and consequently $(\bar{x}^TM_\varepsilon y)u^Te\leq(1+\varepsilon)u^Te\leq(1+\varepsilon)u^TRy$ for any probability vector u. This gives $\bar{x}^TM_\varepsilon y\leq(1+\varepsilon)u^TRy$, i.e., $\bar{x}^TM_\varepsilon\bar{y}\leq(1+\varepsilon)u^TR\bar{y}$. Similarly, for any arbitrary probability vector v and any column representative S of $\mathscr{B}$ we have $\bar{x}^TN_\varepsilon\bar{y}\leq(1+\varepsilon)\bar{x}^TSv$. Thus we have (2).

In order to see that (3) is a reformulation of (2), we use the inequalities

$$\left(1+\frac{\varepsilon}{L}\right)u^TPv\leq u^TPv+\varepsilon\leq\left(1+\frac{\varepsilon}{l}\right)u^TPv,$$

where P is any row/column representative of either $\mathscr{A}$ or $\mathscr{B}$, u and v are probability vectors, l and L are, respectively, the (positive) lower and upper bounds of entries in matrices of $\mathscr{A}$ and $\mathscr{B}$.

Now for the converse. Suppose that the pair $(\bar{x},\bar{y})$ of probability vectors satisfies the conclusion of Theorem 1. Since $\{M_\varepsilon\}$ and $\{N_\varepsilon\}$ are uniformly bounded (with M_ε and N_ε as in (3)), we can let $M_\varepsilon\to M\in\mathbb{R}^{m\times n}$, $N_\varepsilon\to N\in\mathbb{R}^{m\times n}$ as $\varepsilon\downarrow 0$. Then, from (3)

$$\bar{x}^TM\bar{y}\leq u^TR\bar{y}\quad\text{and}\quad\bar{x}^TN\bar{y}\leq\bar{x}^TSv$$

where u,v,R, and S are as in Theorem 1. This implies

$$(\bar{x}^TM\bar{y})e\leq R\bar{y}\quad\text{and}\quad(\bar{x}^TN\bar{y})e\leq S^T\bar{x}.$$

Since M and N are positive, we can put $x^*=\bar{x}/(\bar{x}^TN\bar{y})$, $y^*=\bar{y}/(\bar{x}^TM\bar{y})$ and get

$$e\leq Ry^*\quad\text{and}\quad e\leq S^Tx^*\tag{11}$$

for any row representative matrix R of $\mathscr{A}$, and for any column representative matrix S of $\mathscr{B}$. Furthermore, $x^{*T}(My^*-e)=0$ and $y^{*T}(N^Tx^*-e)=0$. Since x^*,y^*,My^*-e, and N^Tx^*-e are all nonnegative vectors, we conclude that $x^*\wedge(My^*-e)=0$ and $y^*\wedge(N^Tx^*-e)=0$. From (11), $\inf_A(Ay^*-e)\geq 0$. Letting $\varepsilon\downarrow 0$ in the inequality $0\leq x^{*T}\{\inf_A(Ay^*-e)\}\leq x^{*T}[M_\varepsilon y^*-e]$ we see that $x^{*T}\{\inf_A(Ay^*-e)\}=0$ so that $x^*\wedge\inf_A(Ay^*-e)=0$. Similarly, $y^*\wedge\inf_B(B^Tx^*-e)=0$. Thus we have $F(z^*)=0$. □

Our next task is to show that the equation $F(z) = 0$ has a solution. We carry out this task in the following result by employing degree theoretic arguments.

Proposition 2: Assume that $\mathscr{A}$ *and* $\mathscr{B}$ are two nonempty sets of matrices in $\mathbb{R}^{m \times n}$ which are (componentwise) uniformly bounded below by the positive number l and above by the positive number L. *Let* $p \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$ be positive. Then

(a) $F_{(p,q)}$ is continuous on $\mathbb{R}^m \times \mathbb{R}^n$ $(= \mathbb{R}^{m+n})$;
(b) The solution set of the equation $F_{(p,q)}(z) = 0$ is bounded. In fact, for any solution $z = (x, y)$, we have

$$\| x \| + \| y \| \leq l^{-1}(\| p \| + \| q \|);$$

(c) For the function $F\ (= F_{(e,e)})$ there is a bounded set Ω in $\mathbb{R}^{m+n}$ such that

$$\deg(F, \Omega, 0) = -1.$$

In particular, the complementarity system $F(z) = 0$ has a solution.

Proof: (*a*) It is enough to show that the functions ϕ_p and ψ_q are continuous. We demonstrate the continuity of $\phi := \phi_p$. For any index $i \in \{1, 2, \ldots, m\}$,

$$\begin{aligned}[\phi(z)]_i &= \inf_{A \in \mathscr{A}}[(Az - p)]_i = \inf_{A \in \mathscr{A}}[Ay - p + A(z - y)]_i \\ &\leq \inf_{A \in \mathscr{A}}[Ay - p]_i + L\| z - y \| = [\phi(y)]_i + L\| z - y \|.\end{aligned}$$

By symmetry, we have $[\phi(y)]_i \leq [\phi(z)]_i + L\| z - y \|$ and hence $|[\phi(y) - \phi(z)]_i| \leq L\| y - z \|$ $(1 \leq i \leq m)$, proving the continuity of ϕ.
(*b*) Consider any zero (x, y) of $F_{(p,q)}$ so that

$$x \wedge \inf_{A \in \mathscr{A}} (Ay - p) = 0 \quad \text{and} \quad y \wedge \inf_{B \in \mathscr{B}} (B^T x - q) = 0.$$

It is clear that x and y are nonnegative; moreover, they are nonzero since p and q are positive. Assume $x_{i_0} > 0$, so that $\inf_{A \in \mathscr{A}}(Ay - p)_{i_0} = 0$. For any $\varepsilon > 0$, there exists some $A \in \mathscr{A}$ such that $0 \leq [Ay - p]_{i_0} < \varepsilon$. Then $(Ay)_{i_0} < (p_{i_0} + \varepsilon)$, hence $l\| y \| < (p_{i_0} + \varepsilon)$, i.e., $\| y \| < l^{-1}(\| p \| + \varepsilon)$. Since ε is arbitrary, we see that $\| y \| \leq l^{-1}\| p \|$. Similarly, $\| x \| \leq l^{-1}\| q \|$. Adding the two inequalities, we get the stated result.
(*c*) Let K be the $m \times n$ matrix in which every entry is one. Let $0 < \alpha \in \mathbb{R}^m$ and $0 < \beta \in \mathbb{R}^n$ such that $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ and $\beta_1 > \beta_2 > \cdots \beta_n$. Now, the function H_t (for $t \in [0, 1]$) defined by

$$H_t(z) = \begin{bmatrix} x \wedge \inf_{A \in \mathscr{A}}\{tAy + (1 - t)Ky - te - (1 - t)\alpha\} \\ y \wedge \inf_{B \in \mathscr{B}}\{tB^T x + (1 - t)K^T x - te - (1 - t)\beta\} \end{bmatrix}$$

sets up a homotopy between $H_1 := F$ and $H_0 := G$. If $z = (x_t, y_t)$ is a solution of the system $H_t(z) = 0$, then by an application of (b), we have $\|x_t\| + \|y_t\| \leq \bar{l}^{-1}(\|\alpha\| + \|\beta\| + m + n)$ where $\bar{l} := \min\{l, 1\}$. This implies that the set

$$E = \{z : H_t(z) = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Now let Ω be the open ball in $\mathbb{R}^{m+n}$ of radius $\bar{l}^{-1}(\|\alpha\| + \|\beta\| + m + n) + 1$. Then Ω contains E and by the homotopy invariance property of the degree,

$$\deg(F, \Omega, 0) = \deg(G, \Omega, 0).$$

We now show that $\deg(G, \Omega, 0) = -1$. We first show that the equation $G(z) = 0$ has a unique solution $\hat{z} = (\hat{x}, \hat{y})$ with $\hat{x} = (\beta_1, 0, 0 \ldots, 0)^T$, $\hat{y} = (\alpha_1, 0, 0, \ldots, 0)^T$. For suppose that $z = (x, y)$ is any solution. Then

$$x \wedge (Ky - \alpha) = 0 \quad \text{and} \quad y \wedge (K^T x - \beta) = 0$$

and hence

$$x \wedge (\|y\| e - \alpha) = 0 \quad \text{and} \quad y \wedge (\|x\| e - \beta) = 0.$$

We see that $\|y\| \geq \alpha_1 > \alpha_2 > \cdots \alpha_m$ and $\|x\| \geq \beta_1 > \beta_2 > \cdots \beta_n$. We conclude from the above complementarity conditions that $x_2 = x_3 = \cdots = x_m = y_2 = y_3 = \cdots = y_n = 0$, and since x and y are both nonzero, $y_1 = \|y\| = \alpha_1$ and $x_1 = \|x\| = \beta_1$. Hence $(x, y) = (\hat{x}, \hat{y})$.

Let

$$Q = \begin{bmatrix} 0 & K \\ K^T & 0 \end{bmatrix}, \; r = \begin{bmatrix} -\alpha \\ -\beta \end{bmatrix}, \quad \text{and} \quad \hat{z} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}.$$

Then $\hat{z}$ is the unique solution of $G(z) = z \wedge (Qz + r) = 0$ and $\hat{z} + Q\hat{z} + r > 0$. From (8),

$$\deg(G, \Omega, 0) = \operatorname{sgn}\det Q_{II} = \operatorname{sgn}\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

At this particular stage, we have proved that $\deg(F, \Omega, 0) = -1$. Since the degree is nonzero, F must have a zero in the open set Ω. This completes the proof of (c). □

4 Existence of an ε-Equilibrium

Proof of Theorem 1: Since the conclusion of the theorem remains the same if the same constant is added to all the matrices in $\mathscr{A}$ and $\mathscr{B}$, without loss of generality we may assume that the matrices under consideration have a uniform positive lower bound. Now the existence of an ε-equilibrium follows from Propositions 1 and 2. Assume now that both sets $\mathscr{A}$ and $\mathscr{B}$ are compact. Since the set of all row (column) representatives of $\mathscr{A}$ (resp., $\mathscr{B}$) is compact in $\mathbb{R}^{m\times n}$, by letting ε go to zero through an appropriate sequence, we may assume that $M_\varepsilon \to M_0$ and $N_\varepsilon \to N_0$, where M_0 and N_0 are, respectively, some row and column representatives of $\mathscr{A}$ and $\mathscr{B}$. Letting $\varepsilon \downarrow 0$ in (3), we get the desired conclusion. □

Remark 1: In the game $\Gamma(\mathscr{A},\mathscr{B})$ and in Theorem 1, the prominent role is played by the sets $\mathscr{R}_i := \{A_i : A \in \mathscr{A}\}$ $(i = 1, 2, \ldots, m)$ and $\mathscr{C}^j := \{B^j : B \in \mathscr{B}\}$ $(j = 1, 2, \ldots, n)$ (which were used to form row and column representatives of $\mathscr{A}$ and $\mathscr{B}$ respectively) and not the actual sets $\mathscr{A}$ and $\mathscr{B}$. This suggests that players can deal with these sets (with possibly different cardinalities) instead of dealing with sets of matrices $\mathscr{A}$ and $\mathscr{B}$.

Remark 2: When $\mathscr{A}$ and $\mathscr{B}$ are singleton sets, Theorem 1 reduces to Nash's theorem on bimatrix games.

Remark 3: In the case of $\mathscr{A} = \{A, C\}$ and $\mathscr{B} = \{B, D\}$ it is interesting to note that the system $F(z) = 0$ is a vertical linear complementarity problem

$$z \wedge (Q_1 z + q_1) \wedge (Q_2 z + q_2) = 0$$

where

$$Q_1 = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & C \\ D^T & 0 \end{bmatrix}, \quad \text{and} \quad q_1 = q_2 = \begin{bmatrix} -e \\ -e \end{bmatrix}. \tag{12}$$

For a detailed analysis of vertical linear complementarity problems, see Gowda and Sznajder (1994).

Remark 4: Corresponding to sets $\mathscr{A}$ and $\mathscr{B}$ in Theorem 1, let

$$K_1(x,y) := -\inf\{x^T R y : R \text{ is a row representative of } \mathscr{A}\},$$
$$K_2(x,y) := -\inf\{x^T S y : S \text{ is a column representative of } \mathscr{B}\}.$$

Then condition (3) yields

$$K_1(\bar{x},\bar{y}) \geq K_1(u,\bar{y}) - \varepsilon \quad \text{and} \quad K_2(\bar{x},\bar{y}) \geq K_2(\bar{x},v) - \varepsilon$$

which, by letting $\varepsilon \downarrow 0$, become

$$K_1(\bar{x},\bar{y}) \geq K_1(u,\bar{y}) \quad \text{and} \quad K_2(\bar{x},\bar{y}) \geq K_2(\bar{x},v)$$

for all probability vectors u and v. It is interesting to note that the above two functions are convex in each variable separately, while the standard equilibrium theorems require concavity in each variable (cf. Theorem 7.2.2 in Parthasarathy and Raghavan (1971)).

5 Stability of a Game

In this section, we demonstrate how degree theory can be used to study stability aspects of the game $\Gamma(\mathscr{A},\mathscr{B})$. The key idea, which works in the setting of an equation, is very simple. Suppose we know that degree of a function f at 0 over a bounded open set $\mathcal{O}$ is nonzero. Then by the properties of degree listed in Section 2, we see that not only the equation $f(x)=0$ has a solution in $\mathcal{O}$, but the equation $g(x)=0$ has a solution in $\mathcal{O}$ where g is any continuous function "close" to f. By appropriately choosing $\mathcal{O}$, we deduce the "stability" of certain subset(s) of the solution set of $f(x)=0$. It is our contention that degree theory often provides elegant and shorter proofs of stability results, and many times, new information. As an illustration, we establish below a generalization of a bimatrix game result due to Wen-tsün and Jia-he (1962) based on essential fixed points; see Jansen (1981) for a proof based on a theorem of Nikaido and Isoda (Ref. 20 in Jansen (1981)).

For our discussion, we assume that $\mathscr{A}$ and $\mathscr{B}$ are compact and let $\mathscr{E}(\mathscr{A},\mathscr{B})$ denote the set of all equilibrium pairs $(\bar{x},\bar{y})$ corresponding to $\Gamma(\mathscr{A},\mathscr{B})$. Let d_H denote the Hausdorff distance between sets, $\mathbb{B}$ the open unit ball in $\mathbb{R}^m\times\mathbb{R}^n$, and $\mathbb{B}_\varepsilon(\bar{x},\bar{y})$ the open ball of radius ε around $(\bar{x},\bar{y})$.

In the game theory literature, there are numerous stability concepts, see van Damme (1991). In the context of $\Gamma(\mathscr{A},\mathscr{B})$ we always have the following "global" stability property which can be deduced by a straightforward use of compactness and Theorem 1:

For every $\varepsilon>0$, there exists $\delta>0$ such that

$$\mathscr{E}(\mathscr{A}',\mathscr{B}')\cap(\mathscr{E}(\mathscr{A},\mathscr{B})+\varepsilon\mathbb{B})\neq\emptyset$$

for all $\mathscr{A}',\mathscr{B}'$ compact in $\mathbb{R}^{m\times n}$, and $d_H(\mathscr{A},\mathscr{A}')+d_H(\mathscr{B},\mathscr{B}')<\delta$.

The above property deals with the entire set $\mathscr{E}(\mathscr{A},\mathscr{B})$. By a routine use of Zorn's Lemma and compactness (as done, e.g., in the proof of Corollary 10.3.2 in van Damme (1991)), we get certain minimal closed subsets of $\mathscr{E}(\mathscr{A},\mathscr{B})$ enjoying the same property. The following definition, describing the "local" stability behavior, deals with singleton subsets of $\mathscr{E}(\mathscr{A},\mathscr{B})$ which are minimal, and is similar to the definition of an essential equilibrium in an n-person normal form game, see van Damme (1991).

Definition 1: An isolated equilibrium pair $(\bar{x}.\bar{y})$ of $\Gamma(\mathscr{A},\mathscr{B})$ is called stable if for every $\varepsilon>0$ there exists a $\delta>0$ such that for each game $\Gamma(\mathscr{A}',\mathscr{B}')$ with $\mathscr{A}'$ and $\mathscr{B}'$

compact in $\mathbb{R}^{m\times n}$, *and* $d_H(\mathscr{A},\mathscr{A}')+d_H(\mathscr{B},\mathscr{B}')<\delta$, we have

$$\mathscr{E}(\mathscr{A}',\mathscr{B}')\cap\mathbb{B}_\varepsilon(\bar{x},\bar{y})\neq\varnothing.$$

We are now ready for our stability result.

Theorem2: If, for compact $\mathscr{A}$ and $\mathscr{B}$, $\Gamma(\mathscr{A},\mathscr{B})$ has finite number of equilibrium pairs, then it has a stable equilibrium pair.

Proof: Assume without loss of generality that $\mathscr{A}$ and $\mathscr{B}$ are sets of positive matrices, and

$$\mathscr{E}(\mathscr{A},\mathscr{B})=\{(\bar{x}_1,\bar{y}_1),(\bar{x}_2,\bar{y}_2),\dots,(\bar{x}_k,\bar{y}_k)\}.$$

For $(x,y)\in\mathscr{E}(\mathscr{A},\mathscr{B})$, let $\mathscr{D}(x,y)=\{(\lambda x,\mu y):\lambda,\mu>0\}$. It is easily seen that

$$\mathscr{D}(x',y')\cap\mathscr{D}(x'',y'')=\varnothing \tag{13}$$

whenever (x',y') and (x'',y'') are distinct pairs in $\mathscr{E}(\mathscr{A},\mathscr{B})$. The proof of Proposition 1 shows that

$$F(x,y)=0\Rightarrow(\bar{x},\bar{y}):=(x/\|x\|,\,y/\|y\|)\in\mathscr{E}(\mathscr{A},\mathscr{B}).$$

Hence $\mathscr{S}:=\{(x,y):F(x,y)=0\}\subseteq\bigcup_{i=1}^k\mathscr{D}(\bar{x}_i,\bar{y}_i)$, and so $\mathscr{S}=\bigcup_{i=1}^k\mathscr{S}_i$, where $\mathscr{S}_i:=\mathscr{S}\cap\mathscr{D}(\bar{x}_i,\bar{y}_i)$. Clearly, $\mathscr{S}_i\cap\mathscr{S}_j=\varnothing$ whenever $i\neq j$. By Proposition 2, each $\mathscr{S}_i$ is bounded. That each $\mathscr{S}_i$ is closed can be seen as follows: let $(x^{(l)},y^{(l)}):=(\lambda_l\bar{x}_i,\mu_l\bar{y}_i)\in\mathscr{S}_i$ and $(x^{(l)},y^{(l)}\to(\bar{x},\bar{y})$. Since $\|x^{(l)}\|=\lambda_l\to\|\bar{x}\|$ and $\|y^{(l)}\|=\mu_l\to\|\bar{y}\|$, we see that $(\bar{x},\bar{y})=(\|\bar{x}\|\bar{x}_i,\ \|\bar{y}\|\bar{y}_i)$. Moreover, $(\bar{x},\bar{y})\in\mathscr{S}$ and hence by (10), $\|\bar{x}\|\neq0\neq\|\bar{y}\|$. Then $(\bar{x},\bar{y})\in\mathscr{S}\cap\mathscr{D}(\bar{x}_i,\bar{y}_i)=\mathscr{S}_i$.

At this stage we know that $\mathscr{S}_i$s are mutually disjoint and compact. Now for a given $\bar{\delta}>0$, we can find $U_1,\dots,U_k$, bounded, open, and disjoint subsets of $\mathbb{R}^m\times\mathbb{R}^n$ such that

$$\mathscr{S}_i\subseteq U_i \quad\text{and}\quad d_H(\mathscr{S}_i,U_i)<\bar{\delta}.$$

By Proposition 2 (*c*) and the excision-addition property of the degree, we have

$$-1=\deg(F,\Omega,0)=\sum_{i=1}^k\deg(F,U_i,0).$$

Hence, for some index, say, $i=1$, $\deg(F,U_1,0)\neq0$.

We now claim that $(\bar{x}_1,\bar{y}_1)$ is a stable equilibrium pair. Let $\varepsilon>0$ be given. We can find an open set V_1 so that $\mathscr{S}_1\subseteq V_1\subseteq U_1$ and

$$\left\{\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right):(x,y)\in V_1\right\}\subseteq\mathbb{B}_\varepsilon(\bar{x}_1,\bar{y}_1)$$

The excision-addition property gives $\deg(F, V_1, 0) = \deg(F, U_1, 0) \neq 0$. We can now find a $\delta > 0$ so that for $\mathscr{A}', \mathscr{B}'$ compact with $d_H(\mathscr{A}, \mathscr{A}') + d_H(\mathscr{B}, \mathscr{B}') < \delta$ we have

$$\sup_{(x,y)\in V_1} \| F(x,y) - F'(x,y) \| < \operatorname{dist}(0, F(\partial V_1)),$$

where $F'(:= F'_{(e,e)})$ corresponds to $(\mathscr{A}', \mathscr{B}')$ (see (9)). By the nearness property of degree, there exists a solution $(x^\circ, y^\circ) \in V_1$ of the equation $F'(x,y) = 0$. Clearly, $(x^\circ/\|x^\circ\|, y^\circ/\|y^\circ\|)$ belongs to $\mathscr{E}(\mathscr{A}', \mathscr{B}')$ and to $\mathbb{B}_\varepsilon(\bar{x}_1, \bar{y}_1)$. This concludes the proof. □

6 Concluding Remarks

In this article, we have proved a generalization of Nash's equilibrium theorem on bimatrix games. Using degree theoretic ideas, existence and stability results were established for a two-person game in which players are allowed to choose row/column representative matrices and probability distributions. The (powerful) degree theoretic ideas are yet to be fully explored in (other) game theoretic settings.

Professor Thomas Armstrong of the University of Maryland Baltimore County has informed us (Armstrong (1993)) that Theorem 1 for finite $\mathscr{A}$ and $\mathscr{B}$ can also be proved via Nash's theorem on bimatrix games. We wish to thank him for communicating this and for other suggestions. Thanks are also due to the referee for his constructive comments.

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