BULLETIN DE L'ACADÉMIE POLONAISE DES SCIENCES
Série des sciences math., astr. et phys. — Vol. XXV, No, 1, 1977

MATHEMATICS
(FUNCTIONAL ANALYSIS)

On Isomorphic Subspaces of $C(T)$ and $C^m(T)$

by

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Presented by Z. CIESIELSKI on June 8, 1976

Summary. Let $C^m(T)$ be the space of all m-times continuously differentiable complex valued functions on the unit circle T. To each infinite sequence of integers $\{n_k\}$ the following subspace is being assigned: the set of all $f \in C^m(T)$ for which the Fourier coefficients $\hat{f}(n)$ are vanishing whenever $n \notin \{n_k\}$. It is proved that the subspaces in $C(T)=C^0(T)$ and $C^m(T)$ corresponding to $\{n_k\}$ are isomorphic. Similar result holds for the disc algebras A and A^m of functions analytic inside, continuous and m-times continuously differentiable on the boundary of the disc respectively.

Let us consider for a given nonnegative integer m the Banach space $C^m(T)$ $(C^0(T) \equiv C(T))$ with the norm

$$_m\|f\| = \sum_{j=0}^{m} \|f^{(j)}\|_\infty .$$

In what follows we denote by $\{n_k\}$ fixed but arbitrary infinite sequence of integers.

The set of all integers is denoted by $\mathbf{Z}$, the set of all positive integers by $\mathbf{N}$ and $\mathbf{N}_+ = \mathbf{N} \cup \{0\}$.

In $C^j(T)$ the following subspaces are defined:

$$K^j(T) = \{f \in C^j(T) : \hat{f}(n) = 0,\ n \notin \{n_k\}\},$$

$$K_0^j(T) = \{f \in K^j(T) : \hat{f}(0) = 0\}, \quad j \in \mathbf{N}_+,$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt, \quad n \in \mathbf{Z}.$$

Since the Fourier coefficients are continuous linear functionals it follows that $K^j(T)$ and $K_0^j(T)$ are Banach spaces.

THEOREM. *The spaces $K^j(T)$ and $K^{j+1}(T)$ are linearly isomorphic* $(j \in \mathbf{N}_+)$.

Proof. Let us define the operator $S : K_0^j(T) \to C^{j+1}(T)$

$$Sf(t) = \int_{-\pi}^{t} f(s)\, ds$$

Let $G : K_0^j(T) \to K_0^{j+1}(T)$ be the periodic integration operator:

$$Gf(t) = Sf(t) - \widehat{Sf}(0).$$

Let us now compute the Fourier coefficients of the function Gf for $f \in K_0^j(T)$. For $n \neq 0$ $\widehat{Gf}(n) = \widehat{Sf}(n)$, and

$$\widehat{Sf}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} \int_{-\pi}^{t} f(s)\, ds\, dt =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \int_{s}^{\pi} e^{-int} dt\, ds = -\frac{1}{in} \left(-\hat{f}(n) + e^{-in\kappa} \hat{f}(0)\right) = 0$$

whenever $n \notin \{n_k\}$, as well $\widehat{Gf}(0) = 0$.

It means that for all $n \notin \{n_k\}$ $\widehat{Gf}(n) = 0$ and therefore $Gf \in K_0^{j+1}(T)$.

The operator G has the following properties:

(a) G is continuous linear operator:

$$_{j+1}\|Gf\| = \sum_{k=0}^{j+1} \|(Gf)^{(k)}\|_\infty = \|Gf\|_\infty + \sum_{k=1}^{j+1} \|f^{(k-1)}\|_\infty =$$

$$= \|Gf\|_\infty + {}_j\|f\| \leqslant 4\pi \|f\|_\infty + {}_j\|f\| \leqslant (4\pi + 1)\, {}_j\|f\|,$$

whence $\|G\| \leqslant 4\pi + 1$.

(b) G is monomorphism.

(c) G is epimorphism:

If $F \in K_0^{j+1}(T)$ then $F' \in C^j(T)$ and in addition

$$\widehat{F'}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F'(t) e^{-int} dt = (in) \hat{F}(n)$$

for $n \notin \{n_k\}$ and $n \neq 0$, as well as $\widehat{F'}(0) = \dfrac{F(\pi) - F(-\pi)}{2\pi} = 0$.

This implies that $F' \in K_0^j(T)$, but $F = GF'$, i.e. G is inverse to differentiation.

Using the Banach theorem on inverse operators, we find that $G: K_0^j(T) \to K_0^{j+1}(T)$ is an isomorphism.

Now, if $0 \notin \{n_k\}$ then $K_0^j(T) = K^j(T)$ and $K_0^{j+1}(T) = K^{j+1}(T)$ so $K^j(T)$ and $K^{j+1}(T)$ are isomorphic. If $0 \in \{n_k\}$, then $K_0^j(T) \oplus \mathbf{C} \approx K^j(T)$ and $K_0^{j+1}(T) \oplus \mathbf{C} \approx K^{j+1}(T)$.

Consequently,

$$K^j(T) \approx K_0^j(T) \oplus \mathbf{C} \approx K_0^{j+1}(T) \oplus \mathbf{C} \approx K^{j+1}(T),$$

isomorphism $I: K^j(T) \to K^{j+1}(T)$ is given by the formula:

$$If(t) = G\left(f - \hat{f}(0)\right)(t) + \hat{f}(0).$$

COROLLARY 1. *For given $m \in \mathbf{N}_+$ we have $K(T) \equiv K^0(T) \approx K^m(T)$.*

By the Theorem the m-th iteration I^m of I gives the required isomorphism.

The next corollary is preceded by the following remarks. According to the maximum modulus principle, $A(D)$ can be identified with $A(T)$ the space of all $f \in C(T)$

which are boundary functions for the functions from $A(D)$. If follows now by a known result (cf. [4], Theorem 2.2.1) that

$$A(T)=\{f\in C(T): \hat{f}(n)=0,\ n<0\}.$$

We also introduce the space

$$A^m(D)=\{f: f^{(k)}\in A(D),\ k=0,1,\dots,m\},$$

where $f^{(k)}$ is the k-th complex derivative of f.

In $A^m(D)$ the following norm is given

$$_m\|f\|=\sum_{j=0}^{m}\|f^{(j)}\|_\infty .$$

Now if we consider

$$A^m(T)=\{f: f=g|_T,\ g\in A^m(D)\}$$

with the norm induced from $C^m(T)$, then by [2] (Theorem 3.11)

$$A^m(D)\approx A^m(T).$$

Moreover by the results quoted above

$$A^m(T)=\{f\in C^m(T): \hat{f}(n)=0 \text{ for } n<0\}.$$

Let $\{n_k\}=\mathbf{N}_+$, then by these remarks and by Corollary 1 we get

COROLLARY 2. *The spaces* $A(T)=A(D)$, $A^m(T)$ *and* $A^m(D)$ *are isomorphic.*

Remark. It is easy to see (by the proof of the Theorem) that the operator $\boldsymbol{L}$ defined on polynomials

$$L(W_n)=L\left(\sum_{k=0}^{n}a_k z^k\right)=a_0+(-i)^m\sum_{k=1}^{n}a_k\cdot\frac{z^k}{k^m}$$

can be extended to the isomorphism from $A(D)$ onto $A^m(D)$.

Since it is known ([1]) that the space $A^*(T)$ is weakly sequentially complete by Corollary 2 we get

COROLLARY 3. *The space* $[A^m(T)]^*$ *is weakly sequentially complete.*

Taking $\{n_k\}=\mathbf{Z}$ we obtain by the Corollary 1 that the spaces $C(T)$ and $C^m(T)$ are isomorphic.

In the case of $n_k=2^k$ $(k\in\mathbf{N})$ by Sidon's theorem ([3], p. 121) and by the Corollary 1 we get

$$K^j(T)\approx l^1,\quad j\in\mathbf{N}_+ .$$

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Р. Шнаидер, **О изоморфных подпространствах в $C(T)$ и $C^m(T)$**

Содержание. Пусть $C^m(T)$ — пространство всех m-раз непрерывно дифференцируемых комплексных функций на единичной окружности T. Для каждой бесконечной последовательности целых чисел $\{n_k\}$ рассматривается подпространсцво всех $f \in C^m(T)$, для которых коэффициенты Фурье $\hat{f}(n)$ обращаются в нуль при $n \notin \{n_k\}$. Доказано, что все подпространства в $C(T) \equiv C^0(T)$ и в $C^m(T)$, соответствующие одной и той же последовательности $\{n_k\}$, изоморфны. Аналогичный результат имеет место для алгебр A и A^m функций аналитических внутри единичного круга и непрерывных (соответственно m-раз непрерывно дифференцируемых) на границе круга.