ON THE LIPSCHITZ CONTINUITY OF THE SOLUTION MAP IN SOME GENERALIZED LINEAR COMPLEMENTARITY PROBLEMS

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Abstract: *This paper investigates the Lipschitz continuity of the solution map in the settings of horizontal, vertical, and mixed linear complementarity problems. In each of these cases, we show that the solution map is (globally) Lipschitzian if and only if the solution map is single-valued. These generalize a similar result of Murthy, Parthasarathy, and Sabatini proved in the LCP setting.*

1 Introduction

This paper is a continuation of our recent efforts to understand the Lipschitzian behavior of the solution map arising from piecewise affine equations. For the linear complementarity problem (LCP), see Section 4, corresponding to a matrix $M \in \mathbb{R}^{n\times n}$, Murthy, Parthasarathy, and Sabatini [8] have shown that the solution mapping

$$q \mapsto S(q) := \{x : x \geq 0,\, Mx + q \geq 0, \quad \text{and} \quad x^T(Mx+q) = 0\}$$

is (globally) Lipschitzian on $\mathbb{R}^n$ if and only if S is single-valued (equivalently, M is a P-matrix). The main aim of this paper is to show that a similar result is valid in the contexts of horizontal, vertical, and mixed linear complementarity problems, see Section 4 for definitions. Unlike [8] (where the analysis, though elementary, is based on LCP ideas), our approach is via piecewise affine functions. In [6] Gowda and Sznajder showed that a piecewise affine function $f : \mathbb{R}^n \to \mathbb{R}^n$ is surjective and the inverse map f^{-1} is Lipschitzian on $\mathbb{R}^n$ if and only if f is open (or equivalently, coherently oriented); moreover, when the branching number of f is less than or equal to four, these conditions are equivalent to f being a homeomorphism. While this result can be immediately applied to the LCP (via the mapping $f(x) := Mx^+ - x^-$) and more generally to the affine variational inequality problem (AVI) (via the normal map) [6], it cannot be applied directly to the horizontal, vertical, and mixed LCPs. However, as we see below, simple transformations will allow us to rewrite these problems as piecewise affine equations where the above result could be applied.

2 Preliminaries

Throughout this paper, $\mathbb{B}$ denotes the closed unit ball in the space under consideration. We define $x \wedge y$, $x \vee y$, and $\langle x, y\rangle (= x^T y)$ as, respectively, the componentwise minimum, componentwise maximum, and the usual inner product of vectors x and y. Also, $x^+ := x \vee 0$ and $x^- := (-x) \vee 0$.

For a comprehensive treatment of piecewise affine functions, see [2] or [13]. Formally, a continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *piecewise affine* if there exists a set of triples

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(Ω_j, A_j, a_j) $(j = 1, 2, \ldots, K)$ such that each Ω_j is a polyhedral set in $\mathbb{R}^n$ with nonempty interior, $A_j \in \mathbb{R}^{m\times n}$, $a_j \in \mathbb{R}^m$, and

(a) $\mathbb{R}^n = \cup_{i=1}^K \Omega_i$;

(b) For $i \neq j$, $\Omega_i \cap \Omega_j$ is either empty or a proper common face of Ω_i and Ω_j. In particular, $\text{int } \Omega_i \cap \text{int } \Omega_j = \emptyset$ for $i \neq j$;

(c) $f(x) = A_i x + a_i$ for $x \in \Omega_i$, $i = 1, 2, \ldots, K$.

We shall refer to A_i $(i = 1, 2, \ldots, K)$ as the matrices of f (or matrices defining f). The collection $\{\Omega_i,\, i = 1, 2, \ldots, K\}$ is said to be a *polyhedral subdivision of* $\mathbb{R}^n$ corresponding to f.
The *branching number* of this polyhedral subdivision (or simply that of f) is the maximal number of Ωs that have a common face of dimension $(n-2)$.

When $m = n$, we say that f is *coherently oriented* if all the (square) matrices corresponding to f have the same nonzero determinantal sign.

Piecewise affine functions can also be described equivalently [13] as follows. A continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ is piecewise affine if there exist affine functions $f_1, f_2, \ldots, f_J$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$f(x) \in \{f_1(x), f_2(x), \ldots, f_J(x)\} \quad \text{for all} \quad x \in \mathbb{R}^n.$$

This formulation is particularly useful in studying examples.

We shall say that a multivalued function $G : \mathbb{R}^m \to \mathbb{R}^n$ with the domain $\operatorname{dom} G$ is *Lipschitzian* if there exists a positive number γ such that

$$G(y) \subseteq G(z) + \gamma \|y - z\| \boldsymbol{B} \quad \text{for all} \quad y, z \in \operatorname{dom} G,$$

The above condition implies that G is *lower semicontinuous* on $\operatorname{dom} G$ where we define lower semicontinuity of G on a set $Y \subseteq \operatorname{dom} G$ as follows: for each sequence $\{y^k\}$ in Y converging to $\bar{y} \in Y$, and for any $\bar{x} \in G(\bar{y})$, there exists a sequence $\{x^k\}$ in $\operatorname{ran} G$ such that $x^k \in G(y^k)$ for each k and $\{x^k\}$ converges to $\bar{x}$. When G is polyhedral (that is, the graph of G is a finite union of polyhedral sets) whose domain is convex (or more generally, Lipschitz path-connected), lower semicontinuity turns out to be equivalent to the Lipschitzian property [7]. With specific applications in mind, we shall restrict our attention to the case when G is the inverse of a piecewise affine function. The following results from [7] are crucial for our analysis.

Theorem 1 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is piecewise affine and the range of f has nonempty interior. If f^{-1} is lower semicontinuous on the range of f, then the matrices corresponding to f are nonsingular.*

Theorem 2 *Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise affine function. Then the following conditions are equivalent:*

(a) f is surjective and f^{-1} is lower semicontinuous on $\mathbb{R}^n$.

(b) f is surjective and f^{-1} is Lipschitzian on $\mathbb{R}^n$.

(c) f is coherently oriented.

Moreover, when the branching number of f is less than or equal to four, these conditions are equivalent to *(d) f is a homeomorphism.*

We should note here that a piecewise affine function from $\mathbb{R}^n$ into itself is a homeomorphism if and only if it is injective, and coherently oriented if and only if it is an open map, see Thm. 2.3.1 and Prop. 2.3.1 in [13]. Also, the equivalence of (c) and (d) holds under conditions (involving the so called *k-th branching number*) weaker than what is stated here, see [13] Thm. 2.3.7.

3 The main result

We see from the equivalence of (a) and (d) in Theorem 2 that lower semicontinuity of f^{-1} on all of $\mathbb{R}^n$ guarantees the unique solvability of the equation $f(x) = q$ for all $q \in \mathbb{R}^n$. We may ask whether such a result is valid if we replace $\mathbb{R}^n$ by a subspace of $\mathbb{R}^n$. To be precise, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be piecewise affine, Y be a subspace of $\mathbb{R}^n$, $f^{-1}(q) \neq \emptyset$ for all $q \in Y$, and f^{-1} is lower semicontinuous on Y. Does it follow that $f(x) = q$ has a unique solution for all $q \in Y$? Even under the branching number condition, this question does not seem to have a simple and clearcut answer. The Lipschitzian behavior of the solution map arising in horizontal, vertical, and mixed linear complementarity problems is related to this question. Fortunately, the extra structure available in the formulations of these problems allows us to apply Theorem 2 in an appropriate way.

We now present our main result. Applications of this to various complementarity problems will be discussed in the next section.

Let $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ be a function with the following properties:

(a) ψ is piecewise affine, onto,

(b) branching number of ψ is less than or equal to four, and

(c) there exist matrices $P \in \mathbb{R}^{n\times k}$ and $Q \in \mathbb{R}^{m\times k}$ such that

$$\psi(x,y) = r \iff \psi(x - Pr, y - Qr) = 0 \quad \text{for every } r.$$

A simple example of such a function is $\psi(x,y) = x \wedge y$.

Now consider the piecewise affine function $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l \times \mathbb{R}^k$ defined by

$$H(x,y) = \begin{pmatrix} Mx + Ny \\ \psi(x,y) \end{pmatrix} \tag{1}$$

where $M \in \mathbb{R}^{l\times n}$, $N \in \mathbb{R}^{l\times m}$. It is clear that H is piecewise affine and the branching number of H is less than or equal to four.

For a given $q \in \mathbb{R}^l$, we consider the equation

$$H(x,y) = \begin{pmatrix} q \\ 0 \end{pmatrix}$$

and let $\mathcal{S}(q)$ denote the solution set of this equation. We have the following result characterizing the Lipschitzian behavior of $\mathcal{S}$.

Theorem 3 *Consider the above H with $n+m=k+l$. Then the following are equivalent:*

(i) $\mathcal{S}(q) \neq \emptyset$ *for all* $q \in \mathbb{R}^l$ *and the map* $q \mapsto \mathcal{S}(q)$ *is Lipschitzian on* $\mathbb{R}^l$.

(ii) $\mathcal{S}(q) \neq \emptyset$ *for all* $q \in \mathbb{R}^l$ *and the map* $q \mapsto \mathcal{S}(q)$ *is lower semicontinuous on* $\mathbb{R}^l$.

(iii) $|\mathcal{S}(q)| = 1$ *for all* $q \in \mathbb{R}^l$.

(iv) H *is coherently oriented.*

Proof. The implication $(i) \Longrightarrow (ii)$ is obvious. Assume (ii). For any $q \in \mathbb{R}^l$ and $r \in \mathbb{R}^k$, it follows easily from property (c) of ψ that

$$H^{-1}\begin{pmatrix} q \\ r \end{pmatrix} = (Pr, Qr) + \mathcal{S}(q - MPr - NQr). \tag{2}$$

¿From this equality we easily verify that the piecewise affine function H is onto and H^{-1} is lower semicontinuous. Since the branching number of H is less than or equal to four, from Theorem 2, we see that H is a homeomorphism. By putting $r = 0$ in the above equality, we see that $|\mathcal{S}(q)| = 1$ for all $q \in \mathbb{R}^l$. This is (iii). Now suppose (iii) holds. Then $|\mathcal{S}(q - MPr - NQr)| = 1$ for all q and r. By the equality (2), H is one-to-one, i.e., it is a homeomorphism. By Theorem 2, H is coherently oriented, thus proving (iv). Finally when (iv) holds, by Theorem 2, H is surjective and H^{-1} is Lipschitzian on $\mathbb{R}^l \times \mathbb{R}^k$. Restricting H^{-1} to $\mathbb{R}^l \times \{0\}$, we see that $\mathcal{S}$ is Lipschitzian on $\mathbb{R}^l$. Thus we have (i). ∎

Theorem 4 *Let $n+m=k+l$. Suppose that $\mathcal{S}(q) \neq \emptyset$ for all q in some open subset $\mathcal{E}$ of $\mathbb{R}^l$. If the mapping $\mathcal{S} : q \mapsto \mathcal{S}(q)$ is lower semicontinuous on the domain of $\mathcal{S}$, then the matrices that define H are all nonsingular.*

Proof. Under the given assumption on $\mathcal{S}$, it follows from (2) that $H^{-1}(p)$ will be nonempty for all p in some open set, moreover H^{-1} is lower semicontinuous on $\operatorname{ran} H$. Now the conclusion follows from Theorem 1. ∎

4 Applications

In this section we specialize the previous two results to horizontal, vertical, and mixed linear complementarity problems.

To begin with, recall that the linear complementarity problem LCP(M, q) [1] is to find a vector x such that

$$x \geq 0, \quad Mx + q \geq 0, \quad \text{and} \quad x^T(Mx+q) = 0 \tag{3}$$

where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. This problem is equivalent to solving the piecewise equation $x \wedge (Mx + q) = 0$ or the piecewise equation $Mx^+ - x^- = -q$.

4.1 The horizontal linear complementarity problem

Given a pair of matrices $A, B \in \mathbb{R}^{m \times n}$ and a vector $q \in \mathbb{R}^m$, the horizontal linear complementarity problem, HLCP(A, B, q) [14], [15] is to find vectors x and y in $\mathbb{R}^n$ such that

$$\begin{aligned} Ax - By &= q \\ x \wedge y &= 0\,. \end{aligned}$$

This problem can be formulated as a piecewise linear equation $H(x,y) = \begin{pmatrix} q \\ 0 \end{pmatrix}$ where

$$H(x,y) = \begin{bmatrix} Ax - By \\ x \wedge y \end{bmatrix}. \tag{4}$$

As before, $\mathcal{S}(q)$ denotes the solution set of $H(x,y) = \begin{pmatrix} q \\ 0 \end{pmatrix}$. Note that this H is like the one given in (1) with $\psi(x,y) = x \wedge y$. Clearly this ψ is piecewise affine, onto, and $\psi(x,y) = r$ implies that $\psi(x-r, y-r) = 0$. The polyhedral subdivision corresponding to this ψ is given by $\{\Omega_\alpha : \alpha \subseteq \{1,\dots,n\}\}$ where

$$\Omega_\alpha = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \ x_\alpha \geq y_\alpha, \ x_\beta \leq y_\beta\} \ \text{ for } \ \alpha \subseteq \{1,\dots,n\} \ \text{ and } \ \beta := \alpha^c.$$

It is easily seen that the branching number of ψ is less than or equal to four. Thus Theorem 3 is applicable.

Theorem 5 *Consider the horizontal LCP corresponding to the matrix pair (A,B). Assume that A and B are square. Then the following are equivalent:*

(a) (A,B) is a Q-pair (that is, for every $q \in \mathbb{R}^n$, $\mathcal{S}(q) \neq \emptyset$) and the solution map $q \mapsto \mathcal{S}(q)$ is Lipschitzian.

(b) (A,B) is a Q-pair and the solution map $q \mapsto \mathcal{S}(q)$ is lower semicontinuous.

(c) $|\mathcal{S}(q)| = 1 \ \ \forall q \in \mathbb{R}^n$.

(d) All the column representative matrices of (A,B) have the same nonzero determinantal sign.

We recall that an $n \times n$ matrix C is a *column representative* of (A,B) if for each j, the jth column of C is either the jth column of A or the jth column of B.

Proof. The equivalence of (a), (b), and (c) follow immediately from Theorem 3. We complete the proof by showing that (d) is nothing but the coherence property of H: on the polyhedral set Ω_α described above,

$$H(x,y) = \begin{bmatrix} Ax - By \\ \begin{pmatrix} y_\alpha \\ x_\beta \end{pmatrix} \end{bmatrix} = \begin{bmatrix} A & -B \\ \begin{bmatrix} 0 & 0 \\ 0 & I_\beta \end{bmatrix} & \begin{bmatrix} I_\alpha & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

By the Schur determinantal formula [1] (p. 76), [10], the determinant of the matrix defining H on Ω_α is $\det[A_{\cdot\alpha} \ B_{\cdot\beta}]$ which is precisely the determinant of the column representative of (A,B) corresponding to the index set α. Thus the coherence property of H is condition (d) of the theorem. ∎

Some comments regarding the above theorem are in order. The above result can also be derived using Theorem 19 in [15] by reducing the HLCP problem to the classical linear complementarity problem and then applying the theorem of Murthy, Parthasarathy, and Sabatini [8] mentioned in the Introduction. At the same time, it is possible to deduce this result of Murthy, Parthasarathy, and Sabatini from Theorem 5. We shall omit the details.

At this stage, one may ask whether Theorem 5 is valid for non square matrices. It is known that uniqueness can be achieved in the HLCP only when A and B are square [3]. How about the Lipschitzian property of the solution map? The following proposition and example pertain to this question.

Proposition 1 *Assume that (A,B) is a Q-pair where $A,B \in R^{m\times n}$ and the solution map $q \mapsto \mathcal{S}(q)$ is Lipschitzian. Then $m \le n$.*

Proof. Suppose, if possible, that $m > n$. Then HLCP(A,B,q) can be written as

$$\begin{aligned} x \wedge y &= 0 \\ A_1x - B_1y &= q_1 \\ A_2x - B_2y &= q_2 \end{aligned}$$

where $A_1, B_1 \in \mathbb{R}^{n\times n}$, $A_2, B_2 \in \mathbb{R}^{(m-n)\times n}, q_1 \in \mathbb{R}^n$, and $q_2 \in \mathbb{R}^{m-n}$. Obviously, (A_1,B_1) is a Q-pair. Let $(x^*,y^*) \in \mathcal{S}(A_1,B_1,q_1)$. Since the solution map for the pair (A,B) is Lipschitzian, we have

$$(x^*,y^*) \in \mathcal{S}\left(A,B,\begin{pmatrix} q_1 \\ A_2x^* - B_2y^* \end{pmatrix}\right) \subseteq$$

$$\mathcal{S}\left(A,B,\begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix}\right) + \gamma\left\|\begin{pmatrix} q_1 \\ A_2x^* - B_2y^* \end{pmatrix} - \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix}\right\| \mathbb{B}.$$

Now let $\bar{q}_1$ be arbitrary and $\bar{q}_2 = A_2x^* - B_2y^*$. Then $(x^*,y^*) \in \mathcal{S}(A_1,B_1,\bar{q}_1) + \gamma\|q_1 - \bar{q}_1\|\mathbb{B}$. It follows that the solution map $\bar{q}_1 \mapsto \mathcal{S}(A_1,B_1,\bar{q}_1)$ is Lipschitzian and hence Theorem 5 shows that the problem HLCP$(A_1,B_1,\bar{q}_1)$ has a unique solution. For a given $\bar{q}_1 \in \mathbb{R}^n$, take $q_2 \ne A_2x^* - B_2y^*$ with $(x^*,y^*) \in \mathcal{S}(A_1,B_1,\bar{q}_1)$. Then, HLCP $\left(A,B,\begin{pmatrix} \bar{q}_1 \\ q_2 \end{pmatrix}\right)$ has no solution, contradicting the assumption that (A,B) is a Q-pair. Hence $m \le n$. ■

In the following example, m is less than n, the matrices A and B form a Q-pair, and the solution map is Lipschitzian, yet the corresponding HLCP has more than one solution.

Example. Let

$$\begin{aligned} [I\ 0]x - [I\ 0]y &= q \\ x \wedge y &= 0 \end{aligned}$$

where I denotes the $m \times m$ identity matrix, x and y are in $\mathbb{R}^n$. An easy inspection shows that $\forall q \in \mathbb{R}^m$,

$$\mathcal{S}(q) = \big((q^+,0),(q^-,0)\big) + L$$

where $L := \{((0,u),(0,v)) : u \wedge v = 0\}$. Evidently, $([I\ 0], [I\ 0])$ is a Q-pair, and the corresponding solution map is Lipschitzian, yet $|\mathcal{S}(q)| > 1$.

Here is an application of Theorem 4.

Theorem 6 *Let $n = m$. Suppose that* HLCP(A,B,q) *has nonempty solution set for every $q \in \mathcal{E} \subseteq \mathbb{R}^n$ with* int $\mathcal{E} \ne \emptyset$. *Also, assume that the solution map $q \mapsto \mathcal{S}(q)$ is lower semicontinuous on the domain of $\mathcal{S}$. Then all column representative matrices of (A,B) are nonsingular.*

Proof. We saw in the proof of Theorem 5 that the determinants of column representative matrices of (A,B) are nothing but the determinants of the matrices defining H (given by (4)). The equality (2) shows that H^{-1} is lower semicontinuous on the range of H. The same equality shows that if $\bar{q} \in$ int $\mathcal{E}$, then $\begin{pmatrix} \bar{q} \\ 0 \end{pmatrix}$ belongs to the interior of the range of H. To complete the proof, we need only quote Theorem 1. ■

4.2 The vertical linear complementarity problem

Given

$$\mathbf{M} = (M_1, M_2, \ldots, M_k) \quad \text{and} \quad \mathbf{q} = (q_1, q_2, \ldots, q_k),$$

where each M_j is an $n \times n$ matrix and q_j is an n-vector, the VLCP$(\mathbf{M}, \mathbf{q})$ [5], [14], [15] is to solve the piecewise affine equation

$$(M_1 x + q_1) \wedge (M_2 x + q_2) \wedge \cdots \wedge (M_k x + q_k) = 0. \tag{5}$$

We shall write $\mathbf{\Phi}(\mathbf{q})$ for the solution set of this equation.

By introducing the variables $y^j = M_j x + q_j$, we can write the above equation as

$$F(x, y^1, \ldots, y^k) = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \\ 0 \end{pmatrix} \tag{6}$$

where

$$F(x, y^1, y^2, \ldots, y^k) := \begin{bmatrix} y^1 - M_1 x \\ y^2 - M_2 x \\ \vdots \\ y^k - M_k x \\ y^1 \wedge y^2 \ldots \wedge y^k \end{bmatrix}$$

with y^j denoting the jth vector (and not the jth coordinate).

Let $\mathcal{S}(\mathbf{q})$ denote the solution set of (6). Note that the mappings $\mathbf{\Phi}$, $\mathcal{S}$, and F^{-1} have similar lower semicontinuity (Lipschitzian) behavior. This can be easily seen by the equalities

$$F^{-1}\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \\ r \end{pmatrix} = (0, r, r, \ldots, r) + F^{-1}\begin{pmatrix} q_1 - r \\ q_2 - r \\ \vdots \\ q_k - r \\ 0 \end{pmatrix} \tag{7}$$

and

$$F^{-1}\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \\ 0 \end{pmatrix} = \{(x, M_1 x + q_1, M_2 x + q_2, \ldots, M_k x + q_k) : x \in \mathbf{\Phi}(\mathbf{q})\}. \tag{8}$$

For $\boldsymbol{l} = (l_1, \ldots, l_i, \ldots, l_n)$ with $i \in \{1, \ldots, n\}$ and $l_i \in \{1, \ldots, k\}$, we put

$$\Omega_{\boldsymbol{l}} = \bigcap_{i=1}^{n} \bigcap_{j \neq l_i} \{ (x, y^1, \ldots, y^k) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n : \ (y^j)_i \geq (y^{l_i})_i \}. \tag{9}$$

Certainly, $\{\Omega_{\boldsymbol{l}}\}$ forms a polyhedral subdivision associated with the piecewise linear mapping F. With $y = (y^1, \ldots, y^k)$, and $\psi(x, y) := y^1 \wedge y^2 \wedge \ldots \wedge y^k$, the above F looks like H defined in (1). Since ψ has branching number less than or equal to four, Theorem 3 is applicable.

Theorem 7 *Consider the vertical LCP corresponding to* $\mathbf{M}$. *Then the following are equivalent:*

(a) $\mathbf{M}$ *is of* type Q *(that is, for every* $\mathbf{q} \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$, $\Phi(\mathbf{q}) \neq \emptyset$*), and the solution map* $\mathbf{q} \mapsto \Phi(\mathbf{q})$ *is Lipschitzian.*

(b) $\mathbf{M}$ *is of* type Q, *and the solution map* $\mathbf{q} \mapsto \Phi(\mathbf{q})$ *is lower semicontinuous.*

(c) $|\Phi(\mathbf{q})| = 1$ *for all* $\mathbf{q} \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$.

(d) All row representative matrices of $\mathbf{M}$ *have the same nonzero determinantal sign.*

The equivalence of the first three items follows (via the mapping F) immediately from Theorem 3. Only item (d) requires an explanation. By definition, an $n \times n$ matrix C is a *row representative* of $\mathbf{M}$ if for each index j, the jth row of C belongs to the set consisting of jth rows of matrices $M_1, M_2, \ldots, M_k$. It can be shown that the determinant of a row representative of $\mathbf{M}$ is the determinant of a matrix that appears in the piecewise affine formulation of F and conversely. Theorem 3 now gives the equivalence of (c) and (d). The equivalence of (c) and (d) also follows from Theorem 17 in [5].

The following result is an analogue of Theorem 6.

Theorem 8 *Suppose that* $\mathrm{VLCP}(\mathbf{M}, \mathbf{q})$ *has nonempty solution set for every* $\mathbf{q} \in \mathcal{E}$ *with* $\operatorname{int} \mathcal{E} \neq \emptyset$. *If the solution map* $\mathbf{q} \mapsto \Phi(\mathbf{q})$ *is lower semicontinuous on the domain of* Φ, *then all row representative matrices of* $\mathbf{M}$ *are nonsingular.*

Proof. In view of equalities (7) and (8), the lower semicontinuity of Φ implies the lower semicontinuity of F^{-1} on the range of F. To complete the proof, we need only show that the range of F has nonempty interior. This is easily seen since for $\bar{\mathbf{q}} \in \mathcal{E}$, the element $\begin{pmatrix} \bar{\mathbf{q}} \\ 0 \end{pmatrix}$ belongs to the interior of the range of F. ∎

4.3 The mixed linear complementarity problem

Given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$, and vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, the mixed linear complementarity problem [4] is to find vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that

$$\begin{aligned} Ax + By + a &= 0\,, \\ u = Cx + Dy + b\,, \\ u \wedge y &= 0\,. \end{aligned}$$

Let $S(a,b)$ denote the solution set of the above MLCP.

Theorem 9 *The following are equivalent.*

(1) For all $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$, $|S(a,b)| \neq \emptyset$, *and the solution map* $(a,b) \mapsto S(a,b)$ *is Lipschitzian.*

(2) For all $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$, $|S(a,b)| \neq \emptyset$, *and the solution map* $(a,b) \mapsto S(a,b)$ *is lower semicontinuous.*

(3) For all $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$, $|S(a,b)| = 1$.

(4) A *is invertible and* $D - CA^{-1}B$ *is a* $\mathbf{P}$*-matrix.*

Proof. Define the following piecewise affine function

$$F(x,y,u) := \begin{bmatrix} Ax+By \\ u-(Cx+Dy) \\ u \wedge y \end{bmatrix}. \tag{10}$$

Observe that $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ is like H described in (1): for $\alpha \subseteq \{1,\ldots,m\}$ and $\beta := \alpha^c$, define

$$\Omega_\alpha := \{(x,y,u) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m : y_\alpha \geq u_\alpha,\ y_\beta \leq u_\beta\}.$$

The family $\{\Omega_\alpha\}$ forms a polyhedral subdivision of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$. For any $(x,y,u) \in \Omega_\alpha$ we have

$$F(x,y,u) = \begin{bmatrix} Ax+By \\ u-(Cx+Dy) \\ \begin{pmatrix} u_\alpha \\ y_\beta \end{pmatrix} \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ -C & -D & I \\ 0 & E_1 & E_2 \end{bmatrix} \begin{pmatrix} x \\ y \\ u \end{pmatrix} \tag{11}$$

where

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & I_\beta \end{bmatrix} \text{ and } E_2 = \begin{bmatrix} I_\alpha & 0 \\ 0 & 0 \end{bmatrix}.$$

Also,

$$(x^*,y^*) \in \mathcal{S}(a,b) \text{ if and only if } F(x^*,y^*,u^*) = \begin{pmatrix} -a \\ b \\ 0 \end{pmatrix}$$

where $u^* = Cx^* + Dy^* + b$. The equivalence of (*1*), (*2*) and (*3*) follows from Theorem 3. The equivalence of (*3*) and (*4*) is given in Proposition 2 of [11]. ∎

We point out that under the Lipschitzian assumption, Pang [12] proved that matrix A is nonsingular, in which case the MLCP problem can be transformed to the standard LCP, and then we can apply the result of Murthy, Parthasarathy, and Sabatini [8]. Again, our approach is consistent with Theorem 2.

We now state an analogue of Theorem 4.

Theorem 10 *Suppose that* MLCP(A,B,C,D,a,b) *has nonempty solution set for every* $(a,b) \in \mathcal{E} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ *with* int $\mathcal{E} \neq \emptyset$. *Assume also that the solution map* $(a,b) \mapsto \mathcal{S}(a,b)$ *is lower semicontinuous on the domain of* $\mathcal{S}$. *Then* A *is invertible and* $D - CA^{-1}B$ *is nondegenerate (that is, every principal minor is nonzero).*

Let $(a,b) \in \text{int } \mathcal{E}$. It is easily seen that $(-a,b,0)^T \in \text{int ran } F$. Also, F^{-1} is lower semicontinuous on the range of F. By Theorem 1, matrices of F are nonsingular. The lemma below shows how algebraic manipulations involving the Schur determinantal formula lead to the desired conclusion.

Lemma 1 *Let the matrices* A,B,C,D *be as above. Then* A *is invertible and for any index set* $\alpha \subseteq \{1,\ldots,n\}$,

$$\det \begin{bmatrix} A & B & 0 \\ -C & -D & I \\ 0 & E_1 & E_2 \end{bmatrix} = (-1)^m \det A \cdot \det S_{\alpha\alpha}$$

where $S := D - CA^{-1}B$.

Proof of Lemma 1. Let $\varepsilon > 0$ be small, so that $A_\varepsilon := A + \varepsilon I$ is invertible. Then

$$\det \begin{bmatrix} A_\varepsilon & B & 0 \\ -C & -D & I \\ 0 & I & 0 \end{bmatrix} = \det A_\varepsilon \det \begin{bmatrix} -S_\varepsilon & I \\ I & 0 \end{bmatrix} = (-1)^m \det A_\varepsilon$$

by the Schur determinantal formula, where $S_\varepsilon := D - CA_\varepsilon^{-1}B$. Letting $\varepsilon \to 0$, we see that $\det N = (-1)^m \det A$, where

$$N = \begin{bmatrix} A & B & 0 \\ -C & -D & I \\ 0 & I & 0 \end{bmatrix}.$$

Since N is a matrix that appears (for $\alpha = \emptyset$) in the definition of F, we see that $\det A \neq 0$. Now, assume that $\alpha \subseteq \{1, \ldots, n\}$ is arbitrary. Then

$$\det \begin{bmatrix} A & B & 0 \\ -C & -D & I \\ 0 & E_1 & E_2 \end{bmatrix} = \det A \det \begin{bmatrix} -S & I \\ E_1 & E_2 \end{bmatrix} =$$
$$\det A \det (-SE_2 - E_1) = (-1)^m \det A \det (E_1 + SE_2)$$

where the first equality comes from the Schur determinantal formula, the second equality holds because the matrices E_1 and E_2 commute. Also,

$$\det (E_1 + SE_2) = \det \begin{bmatrix} S_{\alpha\alpha} & 0 \\ S_{\beta\alpha} & I \end{bmatrix} = \det S_{\alpha\alpha}.$$

■

5 Concluding Remarks

In this paper we dealt with the global Lipschitzian behavior of the solution map in each of the settings of the horizontal, vertical, and the mixed linear complementarity problem. It is possible to describe the (local) pseudo-Lipschitzian behavior of the solution map at a given solution point for these problems following the results in [7].

References

[1] R.W. Cottle, J.-S. Pang and R.E. stone(1992) *The linear complementarity problem*, Academic Press, Boston .

[2] B.C. Eaves and U.G. Rothblum(1990) *Relationships of properties of piecewise affine maps over ordered fields*, Linear Algebra and Its Applications, 132 pp. 1-63.

[3] M.S. Gowda (1996) *On the extended linear complementarity problem*, Mathematical Programming 72 pp. 33-50.

[4] M.S. Gowda and J.-S. Pang(1994) *Stability analysis of variational inequalities and nonlinear complementarity problems, via the mixed linear complementarity problem and degree theory*, Mathematics of Operations Research, 19 pp. 831-879.

[5] M.S. Gowda and R. Sznajder (1994) *The generalized order linear complementarity problem*, SIAM J. Matrix Analysis and Applications, 15 pp. 779-795.

[6] M.S. GOWDA AND R. SZNAJDER(1996) *On the Lipschitzian properties of polyhedral multifunctions*, Mathematical Programming, 74 pp. 276-278.

[7] M.S. GOWDA AND R. SZNAJDER(1997)*On the pseudo-Lipschitzian behavior of the inverse of a piecewise affine function,* to appear in the Proceedings of the International Conference on Complementarity Problems and Their Applications; M.C. Ferris and J.-S. Pang, eds. SIAM Publications, 1997.

[8] G.S.R. MURTHY, T. PARTHASARATHY, AND M. SABATINI(1996) *Lipschitzian* **Q**-*matriceso are* **P**-*matrices*, Mathematical Programming, 74 .

[9] G.S.R. MURTHY, T. PARTHASARATHY, AND B. SRIPARNA, *Constructive characterization of Lipschitzian* Q_0*-matrices*, Linear Algebra and Its Applications, *forthcoming.*

[10] D.V. OUELLETTE, *Schur complements and Statistics* (1981) Linear Algebra and Its Applications, 36 pp. 187-295.

[11] J.-S. PANG(1990) *Newton's method for B-differentiable equations*, Mathematics of Operations Research, 15 pp. 311-341.

[12] J.-S. PANG (1993) *A degree-theoretic approach to parametric nonsmooth equations with multivalued perturbed solution sets*, Mathematical Programming, 62 pp. 359-383.

[13] S. SCHOLTES (1994) *Introduction to piecewise differentiable equations,* Preprint 53/1994, Institute für Statistik und Mathematische Wirtschaftstheorie, Universität Karlsruhe, 7500 Karlsruhe, Germany, May.

[14] R. SZNAJDER (1994) *Degree-theoretic analysis of the vertical and horizontal linear complementarity problem*, Ph.D. thesis, University of Maryland Baltimore County, Baltimore, Maryland, May.

[15] R. SZNAJDER AND M.S. GOWDA(1995) *Generalizations of* $\mathbf{P}_0$ *and* **P**-*properties; extended vertical and horizontal LCPs*, Linear Algebra and Its Applications, 223/224 pp. 695-715.

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