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On the Lipschitzian properties of polyhedral multifunctions

M. Seetharama Gowda [1,*], Roman Sznajder

Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21228, USA

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Abstract

In this paper, we show that for a polyhedral multifunction $F: R^n \to R^m$ with convex range, the inverse function F^{-1} is locally lower Lipschitzian at every point of the range of F (equivalently Lipschitzian on the range of F) if and only if the function F is open. As a consequence, we show that for a piecewise affine function $f: R^n \to R^n$, f is surjective and f^{-1} is Lipschitzian if and only if f is coherently oriented. An application, via Robinson's normal map formulation, leads to the following result in the context of affine variational inequalities: the solution mapping (as a function of the data vector) is nonempty-valued and Lipschitzian on the entire space if and only if the solution mapping is single-valued. This extends a recent result of Murthy, Parthasarathy and Sabatini, proved in the setting of linear complementarity problems.

Keywords: Polyhedral multifunction; Lipschitzian; Coherence; Open; Error bound; Affine variational inequality

1. Introduction

Consider a polyhedral multifunction $F: R^n \to R^m$, i.e., F is a point-to-set-valued function and the graph of F is a finite union of polyhedral sets. The inverse function F^{-1} is a polyhedral multifunction and hence has the *local upper Lipschitzian property* at each point in the range of F [16]. This means that for each y^* in the range of F, there exist a positive number α and a neighborhood V of y^* such that

$$F^{-1}(y) \subseteq F^{-1}(y^*) + \alpha \| y - y^* \| B \quad \text{for all } y \in V,$$

[*] Corresponding author. e-mail: gowda@math.umbc.edu.

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where B denotes the closed unit ball in R^n. The main purpose of this article is to investigate conditions under which F^{-1} has the local lower Lipschitzian and Lipschitzian properties. Recall that a multivalued function $G: R^m \to R^n$ with domain dom G and range ran G is *locally lower Lipschitzian* at $y^* \in$ dom G if there exist a positive number β and a neighborhood U of y^* such that

$$G(y^*) \subseteq G(y) + \beta \| y - y^* \| B \quad \text{for all } y \in U \cap \text{dom } G.$$

G is *Lipschitzian* if there exists a positive number γ such that

$$G(y) \subseteq G(z) + \gamma \| y - z \| B \quad \text{for all } y, z \in \text{dom } G.$$

Our motivation comes from some recent results in the area of linear complementarity problems. Given a matrix $M \in R^{n \times n}$ and a vector $q \in R^n$, the *linear complementarity problem* [2], denoted by LCP(M, q), is to find a vector $x \in R^n$ such that

$$x \geqslant 0, \qquad Mx + q \geqslant 0, \quad \text{and} \quad \langle x, Mx + q \rangle = 0.$$

With M fixed and $\mathscr{S}(q)$ denoting the solution set of LCP(M, q), Mangasarian and Shiau [13] (see also [2]) prove that the solution map $q \mapsto \mathscr{S}(q)$ is Lipschitzian when M is a P-matrix (meaning that all principal minors of M are positive or equivalently $\mathscr{S}(q)$ is a singleton for every $q \in R^n$). Going in the opposite direction, Gowda [6] proves the converse under the assumptions that M is a Q-matrix (which means that $\mathscr{S}(q) \neq \emptyset$ for all $q \in R^n$) and $\mathscr{S}(e)$ is a singleton set for some positive vector $e \in R^n$. In a recent paper [14], Murthy, Parthasarathy, and Sabatini prove the same converse under the weaker assumption that M is a Q-matrix. When LCP(M, q) is formulated as an equation $f(x) + q = 0$ where the piecewise affine function $f: R^n \to R^n$ (known as the Minty map) is given by

$$f(x) = Mx^+ - x^-, \tag{1}$$

the above results say (see Section 5 for details) that f is surjective and f^{-1} is Lipschitzian if and only if f is a homeomorphism. Because of its special nature, f is a homeomorphism if and only if it is coherently oriented (meaning that matrices associated with f have the same nonzero determinantal sign, see Section 2); since the dimensions of the domain and range of the (piecewise affine function) f coincide, this coherent orientedness is equivalent to openness of f. We thus have: for the Minty map given by (1), f is surjective and f^{-1} is Lipschitzian if and only if f is an open map. Our main theorem (see Section 3) proves such a result for an arbitrary polyhedral multifunction F from R^n into R^m. Actually, we prove the openness of F under the (weaker) assumption that F is surjective and F^{-1} is locally lower Lipschitzian at each point of the range of F.

As an application of our main theorem, we consider affine variational inequalities. Given a matrix $M \in R^{n \times n}$, a polyhedral set $\mathscr{K} \subseteq R^n$, and a vector $q \in R^n$, the *affine variational inequality problem*, denoted by AVI$(M, \mathscr{K}, q)$, is to find a vector $x^* \in \mathscr{K}$ such that

$$\langle Mx^* + q, x - x^* \rangle \geqslant 0 \quad \text{for all } x \in \mathscr{K}. \tag{2}$$

Denoting the solution set of this problem by $\mathscr{S}(q)$ (with fixed M and $\mathscr{K}$), we prove via Robinson's normal map

$$f(x) := M\Pi_{\mathscr{K}}(x) + x - \Pi_{\mathscr{K}}(x) \tag{3}$$

that AVI(M, $\mathscr{K}$, q) has a unique solution for every $q \in R^n$ if and only if $\mathscr{S}(q) \neq \emptyset$ for all $q \in R^n$ and the solution map $q \mapsto \mathscr{S}(q)$ is Lipschitzian.

2. Preliminaries

Throughout this paper, we shall denote the inner product of two vectors x and y by $\langle x, y \rangle$. B denotes the closed unit ball in the space under consideration. For $x \in R^n$, $x^+ := \max\{x, 0\}$ and $x^- := x^+ - x$. The 2-norm of a vector x is denoted by $\|x\|$. The symbol $\Pi_{\mathscr{K}}(x)$ denotes the usual 2-norm projection of x onto the polyhedral set $\mathscr{K}$. Given two vectors p^* and q^* in R^m, we shall say that $\{p^* = p^0, p^1, p^2, \ldots, p^l = q^*\}$ is a partition of the line segment $[p^*, q^*]$ if $p^j \in [p^*, q^*]$ for each j and $\|p^0 - p^j\|$ increases strictly as j goes from 0 to l.

For a multifunction F with domain in R^n and range in R^m, we let $F(x)$ denote the image of the point $x \in R^n$; note that $y \in F(x)$ if and only if $x \in F^{-1}(y)$. The graph of F consists of all pairs (x, y) with x in the domain of F and $y \in F(x)$. Now let F be a polyhedral multifunction whose graph is given by

$$\text{Graph } F = \bigcup_{j=1}^{L} G_j, \tag{4}$$

where each G_j is a (nonempty) polyhedral set in $R^n \times R^m$. (Recall that a polyhedral set is the intersection of a finite number of half-spaces.) Writing Π_1 for the projection of $R^n \times R^m$ onto R^n, we see that

$$\text{dom } F = \bigcup_{j=1}^{L} \Pi_1(G_j).$$

Writing Π_2 for the projection of $R^n \times R^m$ onto R^m, we let

$$\{\Pi_2(G_i) : 1 \leqslant i \leqslant L\} = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_T\},$$

where the sets Λ_j are all distinct. We note that each Λ_j is closed, convex, and ran $F = \bigcup_{j=1}^{T} \Lambda_j$.

We shall say that F is *open* if the image of an open set in R^n (equivalently, in the relative topology of dom F) is open in ran F.

A *piecewise affine function* $f: R^n \to R^m$ is a single-valued continuous function with *domain* R^n for which there exists a set of triples (Ω_j, A_j, a_j) $(j = 1, 2, \ldots, K)$ such that each Ω_j is a polyhedral set in R^n with nonempty interior, $A_j \in R^{m \times n}$, $a_j \in R^m$, and

(a) $R^n = \bigcup_{j=1}^{K} \Omega_j$.

(b) For $i \neq j$, $\Omega_i \cap \Omega_j$ is either empty or a proper common face of Ω_i and Ω_j. In particular, int $\Omega_j \cup$ int $\Omega_i = \emptyset$ for $i \neq j$.

(c) $f(x) = A_j x + a_j$ on Ω_j, $j = 1, 2, \ldots, K$.

The collection $\{\Omega_j : j = 1, 2, \ldots, K\}$ is called the polyhedral subdivision of R^n corresponding to f. When $m = n$, we shall say that f is *coherently oriented* if all the (square) matrices A_j have the same nonzero determinantal sign.

For a comprehensive treatment of piecewise affine functions see [4] or [19]. In the latter reference a piecewise affine function is defined, equivalently, as a single-valued continuous function $f: R^n \to R^m$ for which there exist finitely many affine functions $f_j: R^n \to R^m$ such that

$$f(x) \in \{f_1(x), f_2(x), \ldots, f_J(x)\} \quad \text{for all } x \in R^n.$$

The following well known result (see Theorem 2.3.1 and Proposition 2.3.1 in [19]) summarizes the important properties of piecewise affine functions that are needed in this paper.

Theorem 1. *Suppose that $f: R^n \to R^n$ is piecewise affine. Then*

(i) *f is a homeomorphism if and only if it is injective;*

(ii) *f is coherently oriented if and only if f maps open sets in R^n to open sets in R^n.*

In particular, we note that a homeomorphism is necessarily coherently oriented. Also, a coherently oriented function is necessarily surjective, see the proof of Theorem 5 below.

We also need the following result due to Walkup and Wets [23]:

Theorem 2. *Let Ω be a polyhedral set in R^n and $h: R^n \to R^m$ be an affine function. Then the mapping $y \mapsto \Omega \cap h^{-1}(y)$ from $h(\Omega)$ to R^n is Lipschitzian.*

3. Lipschitzian characterizations

We begin with our main theorem.

Theorem 3. *Let F be a polyhedral multifunction with* dom $F \subseteq R^n$ *and* ran $F \subseteq R^m$. *Assume that* ran F *is convex. Then the following are equivalent:*

(a) *F^{-1} is locally lower Lipschitzian at each point of* ran F.

(b) *F is an open map from* dom F *to* ran F.

(c) *F^{-1} is Lipschitzian on* ran F.

Before we present the proof, we make a few remarks. There are several results in the literature describing the Lipschitzian behavior of F^{-1} on the range of F. Robinson [16] observes, via Theorem 2, that when the graph of F is convex, F^{-1} is Lipschitzian on ran F. The Open Mapping Theorem for a surjective closed convex process between two Banach spaces [1] is another well known result of this nature. In many of the situations that we are interested in, e.g., the linear complementarity problems, affine variational inequality problems, etc., the graph is rarely convex, and yet the above result is applicable.

The important part of the proof of this theorem is in showing that F^{-1} is Lipschitzian whenever F is open. This is partly covered in the following technical lemma. This lemma is reminiscent of Katzenelson's algorithm in nonlinear networks [5] and a lemma proved by Mangasarian and Shiau [13] in connection with the Lipschitzian property of the solution map corresponding to an LCP with a P-matrix. For related results, see Lemma 4.1.1 and Proposition 4.1.2 in [19]. The lemma can also be viewed as a finite algorithm for finding an element of $F^{-1}(q^*)$ starting with an arbitrary p^* and $x^* \in F^{-1}(p^*)$.

Lemma 1. *Suppose that the polyhedral multifunction F is open and has convex range. Let G_i ($i = 1, 2, \ldots, L$) and Λ_j ($j = 1, 2, \ldots, T$) be as in Section 2. Then there exists a positive number γ with the following property: given vectors p^* and q^* in* ran F, *and $x^* \in F^{-1}(p^*)$, there exist a partition $\{p^* = p^0, p^1, \ldots, p^l = q^*\}$ of the line segment $[p^*, q^*]$ and a set $\{x^* = x^0, x^1, \ldots, x^l\}$ such that:*

(i) $0 \leqslant l \leqslant T$;

(ii) $x^j \in F^{-1}(p^j)$ *for* $0 \leqslant j \leqslant l$;

(iii) *for each* $j \in \{0, 1, \ldots, l-1\}$, $\{(x^j, p^j), (x^{j+1}, p^{j+1})\}$ *is contained in some* G_i; *and*

(iv) $\| x^j - x^{j+1} \| \leqslant \gamma \| p^j - p^{j+1} \|$ *for* $0 \leqslant j \leqslant l-1$.

Proof. First we describe γ. For each j, the mapping $y \mapsto G_j \cap \Pi_2^{-1}(y)$ is Lipschitzian by Theorem 2. Since Π_1 is nonexpansive, it follows that $y \mapsto \Pi_1[G_j \cap \Pi_2^{-1}(y)]$ is also Lipschitzian; let γ_j denote its Lipschitzian constant. We put $\gamma := \max_j \gamma_j$.

For convenience, throughout this proof, we denote the domain of F (range of F) by X (respectively, Y). We assume that $p^* \neq q^*$; otherwise the lemma is obvious. Let $x^0 := x^*$, $p^0 := p^*$, and Γ_0 be the collection of all G_js containing (x^0, p^0). Since Γ_0 is finite and all the G_js are closed, there exists an open set $U \times V$ in $R^n \times R^m$ such that

$$(x^0, p^0) \in U \times V \cap \text{Graph } F \subseteq \bigcup_{G_j \in \Gamma_0} G_j.$$

This implies that

$$p^0 \in \Pi_2(U \times V \cap \text{Graph } F) \subseteq \bigcup_{G_j \in \Gamma_0} \Pi_2(G_j).$$

We observe that $\Pi_2(U \times V \cap \text{Graph } F) = V \cap F(U \cap X)$ is open in Y by assumption and that the set on the extreme right of the above inclusion is a union of the sets Λ_i. Since Y is convex, the line segment $[p^0, q^*]$ is in Y and so the open line segment (p^0, q^*) must intersect at least one such Λ_i. Among all such Λ_is, we select one which contains the "maximal" subinterval of $[p^0, q^*]$. To make things formal, let t_0 denote the maximum of real numbers t between 0 and 1 with the property that the line segment $[p^0, (1-t)p^0 + tq^*]$ is contained in some Λ_i (which is of the form $\Pi_2(G_j)$ with

$G_j \in \Gamma_0$). This t_0 is positive, and because Λ_is are closed and convex, there is some Λ_i which contains the interval $[p^0, (1-t_0)p^0 + t_0 q^*]$. Let Λ^0 denote such a Λ_i so that for some $G^0 \in \Gamma_0$ we have $\Lambda^0 = \Pi_2(G^0)$. The element $p^1 := (1-t_0)p^0 + t_0 q^* \in \Pi_2(G^0)$ and so the set $G^0 \cap \Pi_2^{-1}(p^1)$ is nonempty. Now obviously, $x^0 \in \Pi_1[G^0 \cap \Pi_2^{-1}(p^0)]$ and the inclusion

$$\Pi_1\left[G^0 \cap \Pi_2^{-1}(p^0)\right] \subseteq \Pi_1\left[G^0 \cap \Pi_2^{-1}(p^1)\right] + \gamma \| p^1 - p^0 \| B$$

shows that there exists an $x^1 \in \Pi_1[G^0 \cap \Pi_2^{-1}(p^1)]$ such that $\| x^0 - x^1 \| \leqslant \gamma \| p^0 - p^1 \|$. Clearly, $(x^1, p^1) \in G^0$ and hence $p^1 \in F(x^1)$. (Of course, we also have $(x^0, p^0) \in G^0$.) If $p^1 = q^*$, we have (i)–(iv) with $l = 1$. If $p^1 \neq q^*$, starting with x^1 and p^1, we repeat the above process to generate Γ_j, t_j, Λ^j, G^j, p^{j+1}, and x^{j+1}, for $j = 1, 2, \ldots$. We have $x^j \in F^{-1}(p^j)$, $\{(x^j, p^j), (x^{j+1}, p^{j+1})\} \subseteq G^j$ and $\| x^j - x^{j+1} \| \leqslant \gamma \| p^j - p^{j+1} \|$ for each j. We now show that this process must stop in at most T steps. Since at each step the number t_j is positive, it follows from convexity that Λ^j is different from all the previous Λ^i $(0 \leqslant i \leqslant j-1)$. Thus at each step a new Λ is generated. Since there are at most T such Λs, the process must terminate in at most T steps. This completes the proof of the lemma. □

Proof of Theorem 3. ((a) ⇒ (b)): Let U be an open set in R^n and $y^* \in F(U)$; let $x^* \in U$ with $y^* \in F(x^*)$. Assume that no neighborhood of y^* (relative to ran F) is contained in $F(U)$. Then there is a sequence $\{y^k\}$ in ran F such that $y^k \notin F(U)$ and $y^k \to y^*$ as $k \to \infty$. Since F^{-1} is (locally) lower Lipschitzian at y^*, we can find $x^k \in F^{-1}(y^k)$ such that for all k, $\| x^* - x^k \| \leqslant \alpha^* \| y^* - y^k \|$ with α^* positive and independent of k. It follows that $x^k \to x^*$ and hence $x^k \in U$ for all large k. For all such k, $y^k \in F(x^k) \subseteq F(U)$ which is a contradiction.

$(b) \Rightarrow (c)$: Assume that F is open. Let γ be as in the above lemma. We show, for any p^* and q^* in ran F, that

$$F^{-1}(p^*) \subseteq F^{-1}(q^*) + \gamma \| p^* - q^* \| B. \tag{5}$$

Let $x^* \in F^{-1}(p^*)$. Applying the above lemma to the present setting, we let $y^* := x^l$. Then $y^* \in F^{-1}(q^*)$ and

$$\| x^* - y^* \| \leqslant \sum_{j=0}^{l-1} \| x^j - x^{j+1} \| \leqslant \sum_{j=0}^{l-1} \gamma \| p^j - p^{j+1} \| = \gamma \| p^* - q^* \|,$$

proving (5). The implication $(c) \Rightarrow (a)$ is obvious. □

With $G = F^{-1}$ the above theorem leads to the following theorem.

Theorem 4. *Suppose G is a polyhedral multifunction with convex domain contained in R^m and range contained in R^n. Then G is continuous (i.e., the inverse image of an open set in R^n under G is open in* dom *G) if and only if G is Lipschitz continuous.*

Our next result deals with piecewise affine functions; the most important part here is the equivalence of (c) and (d). Note that piecewise affine functions are polyhedral and for such functions we always assume that the domain is the entire space under consideration.

Theorem 5. *Consider a piecewise affine function $f: R^n \to R^m$. Then the following are equivalent*:

(a) *f is surjective and f^{-1} is locally lower Lipschitzian at each point of R^m.*

(b) *f is an open map from R^n to R^m, i.e., f maps open sets in R^n to open sets in R^m.*

(c) *f is surjective and f^{-1} is Lipschitzian on R^m.*

Moreover, when $m = n$, these are further equivalent to

(d) *f is coherently oriented.*

Proof. It is clear from Theorem 3 that $(a) \Rightarrow (b)$. The range of f is a finite union of polyhedral sets and hence closed. Hence when (b) holds, the range is both open and closed in R^m. It follows that f is surjective. That f^{-1} is Lipschitz follows from Theorem 3. Thus $(b) \Rightarrow (c)$. The implication $(c) \Rightarrow (a)$ is obvious. When $m = n$, the equivalence of (b) and (d) is precisely part (ii) of Theorem 1. □

In general, the homeomorphism and coherent orientation properties for a piecewise affine function from R^n into itself are not equivalent. For example, the function constructed by Scholtes (see Example 2.3.1 in [19]) is not a homeomorphism but yet coherently oriented. However, in some interesting cases, e.g., the linear complementarity problem and the affine variational inequality problem, the two concepts are equivalent. Our next result deals with such an equivalence under a geometric condition. Here we define the branching number of the polyhedral subdivision corresponding to the function from R^n into itself as the maximal number of polyhedral sets in the subdivision having a common face of dimension $n - 2$.

Theorem 6. *Suppose that $f: R^n \to R^n$ is piecewise affine and the corresponding polyhedral subdivision has branching number less than or equal to* 4. *Then f is a homeomorphism if and only if f is surjective and f^{-1} is Lipschitzian.*

Proof. Under the stated hypothesis on the branching number it follows from a result of Kuhn and Löwen [10] that f is a homeomorphism if and only if f is coherently oriented. The result now follows from the previous theorem. □

All of the previous results were proved under the assumption that the range is convex (and the function is surjective in the case of piecewise affine functions). What happens when the range is not convex (and/or the function is surjective)? The following result gives a partial answer for piecewise affine functions.

Theorem 7. *Suppose that $f: R^n \to R^n$ is piecewise affine and the range of f has nonempty interior. If f^{-1} is locally lower Lipschitzian at each point of the range of f, then the matrices corresponding to f are nonsingular.*

Proof. We assume the description of f given in Section 2. Assume that f^{-1} is locally lower Lipschitzian at each point of the range. If each matrix A_j is singular, then the range, being a union of lower dimensional sets $A_j(\Omega_j)+a_j$, cannot have interior. It follows from the hypothesis that at least one A_j is nonsingular. Consider the collection $\mathscr{E}$ of all matrices which are nonsingular. We have to show that $\mathscr{E}$ is the entire collection of matrices of f. Assume, if possible, that $\mathscr{E}=\{A_1, A_2, \dots, A_s\}$ with $s<K$. Consider the union of Ω_js for $j=1, 2, \dots, s$. This union cannot cover R^n and hence there is a boundary point. This union is closed, has nonempty interior, and so the boundary is $(n-1)$-dimensional [9, Corollary IV.4.2]. There is a boundary point, say y^*, which lies in the relative interior of an $(n-1)$-dimensional face Z of, say, Ω_1. This y^* must belong to some other Ω_i with $i \neq 1$. Since $\Omega_1 \cap \Omega_i$ is a common proper face of Ω_1 and Ω_i, it follows that Z is a face of Ω_i also. The index i must be greater than s, else a neighborhood of y^* would be contained in $\operatorname{int}(\Omega_1 \cup \Omega_i)$, contradicting the choice of the boundary point y^*. We now show that A_i is nonsingular contradicting the maximality of s. Suppose that A_i is singular and assume without loss of generality that $i=s+1$. Since $A_{s+1}(Z)+a_{s+1}=f(Z)=A_1(Z)+a_1$ is $(n-1)$-dimensional and $f(\Omega_{s+1})$ is convex and not n-dimensional, it follows that $f(\Omega_{s+1}) \subseteq \operatorname{aff} f(Z)$ where "aff" is an abbreviation for the affine hull. Since $f(y^*)$ is in the relative interior of $f(Z)$, the above inclusion proves that for some vector e in the interior of Ω_{s+1}, $d := f(e)$ belongs to the relative interior of $f(Z)$. Let $d^k \in f(\operatorname{int} \Omega_1)$ such that $d^k \to d$. Note that $d^k \notin \operatorname{aff} f(Z)$ for all k. Since f^{-1} is locally lower Lipschitzian at d, we have, for some positive β, and all large k,

$$f^{-1}(d) \subseteq f^{-1}(d^k) + \beta \, \| d^k - d \| \, B.$$

Since $e \in f^{-1}(d)$, we must have $x^k \in f^{-1}(d^k)$ such that $\| e - x^k \| \leqslant \beta \, \| d^k - d \|$. It follows that $x^k \to e$. But then $x^k \in \operatorname{int} \Omega_{s+1}$ and hence $d^k \in f(\Omega_{s+1}) \subseteq \operatorname{aff} f(Z)$, a contradiction. Thus s must be equal to K, proving that all matrices of f are nonsingular. □

4. Global error bounds

Consider a polyhedral multifunction $F: R^n \to R^m$. It is easily seen that the local upper Lipschitzian property of F^{-1} at a point $p \in \operatorname{ran} F$ is equivalent to the local error bound property: there exists a positive number α such that

$$\operatorname{dist}(z, F^{-1}(p)) \leqslant \alpha \, \| y - p \|$$

for all $z \in \operatorname{dom} F$ and y in $F(z)$ "close" to p. Also, F^{-1} is Lipschitzian if and only if the *global error bound property* holds: There exists a positive number γ such that

$$\operatorname{dist}(z, F^{-1}(p)) \leqslant \gamma \, \| y - p \| \quad \text{for all } p \in \operatorname{ran} F, \ z \in \operatorname{dom} F, \text{ and } y \in F(z).$$

It follows from Theorem 3 that *when the range of F is convex, the above global error bound property holds if and only if F is open from* dom F *to* ran F. With a view towards applications, we shall state this result explicitly for piecewise affine functions.

Related global error bound results for piecewise affine functions can be found in [7], [11] and [12].

Theorem 8. *Suppose $f: R^n \to R^m$ is piecewise affine. Then f is an open map from R^n into R^m (equivalently, coherently oriented when $n = m$) if and only if f is surjective and there exists a positive number γ such that*

$$\operatorname{dist}(z, f^{-1}(p)) \leqslant \gamma \| f(z) - p \| \tag{6}$$

for all $z \in R^n$ and $p \in R^m$.

5. Applications

In this section, we apply our previous analysis to affine variational inequalities. We recall that corresponding to the matrix $M \in R^{n \times n}$ and a polyhedral set $\mathscr{K} \subseteq R^n$, the normal map is given by

$$f(x) = M \Pi_{\mathscr{K}}(x) + x - \Pi_{\mathscr{K}}(x).$$

Note that this reduces to the Minty map in the context of the LCP (where $\mathscr{K}$ is the nonnegative orthant). Robinson [17] showed that the above normal map is a homeomorphism if and only if it is coherently oriented. This can also be deduced by combining the result of Kuhn and Löwen (mentioned in the proof of Theorem 6) and a result due to Ralph [15] who proved that the branching number of the polyhedral subdivision corresponding to the normal map is less than or equal to 4. (It has been shown recently by Scholtes [21] that the branching number is 4.) In view of these observations, we have the following

Theorem 9. *Consider the normal map f given above. Then f is a homeomorphism if and only if f is surjective and f^{-1} is Lipschitzian.*

We now formulate the above result in terms of the solution sets. For a $q \in R^n$, let $\mathscr{S}(q)$ be the solution set of AVI(M, $\mathscr{K}$, q). This is related to $f^{-1}(-q)$ in the following way:

$$\mathscr{S}(q) = \Pi_{\mathscr{K}}(f^{-1}(-q)) \quad \text{and} \quad f^{-1}(-q) = \{u - Mu - q : u \in \mathscr{S}(q)\}. \tag{7}$$

Theorem 10. *For the affine variational inequality corresponding to M and $\mathscr{K}$ as above, the following are equivalent:*

(i) *For all $q \in R^n$, $\mathscr{S}(q) \neq \emptyset$ and the mapping $q \mapsto \mathscr{S}(q)$ is Lipschitzian on R^n.*
(ii) *For all $q \in R^n$,* AVI(M, $\mathscr{K}$, q) *has a unique solution.*

Proof. Let f denote the normal map. Suppose that (i) holds. Then for some positive number η, we have for all p and q,

$$\mathscr{S}(q) \subseteq \mathscr{S}(p) + \eta \| p - q \| B.$$

We show the Lipschitzian property of f^{-1} by proving the inclusion

$$f^{-1}(-q) \subseteq f^{-1}(-p) + \zeta \, \| p - q \| B, \tag{8}$$

where $\zeta := (\| M \| + 1)\eta + 1$, and $\| M \|$ denotes the norm of the matrix (as a linear operator from R^n into itself). Let $u \in f^{-1}(-q)$. Then $u = \Pi_{\mathscr{X}}(u) - M\Pi_{\mathscr{X}}(u) - q$ and $\Pi_{\mathscr{X}}(u) \in \mathscr{S}(q)$; so there exists a $v \in \mathscr{S}(p)$ such that $\| \Pi_{\mathscr{X}}(u) - v \| \leqslant \eta \| p - q \|$. Let $w = v - Mv - p$. Then $w \in f^{-1}(-p)$ from (7). It follows that

$$\| u - w \| \leqslant \| \Pi_{\mathscr{X}}(u) - M\Pi_{\mathscr{X}}(u) - q - (v - Mv - p) \| \leqslant \zeta \, \| p - q \|.$$

Since $\mathscr{S}(q) \neq \emptyset$ for all q, it follows that f is surjective. By the previous theorem f is a homeomorphism. From (7) we see that $\mathscr{S}(q)$ has exactly one solution for each q. Thus we have (ii). If (ii) holds, then $f^{-1}(-q)$ is a singleton for all $q \in R^n$. Thus f is injective as well as surjective. The single-valued function f^{-1} is piecewise affine and hence is Lipschitzian (see Proposition 2.2.7 in [19] or Theorem 4). Since $\Pi_{\mathscr{X}}$ is nonexpansive, it follows that $q \mapsto \mathscr{S}(q)$ is Lipschitzian. This proves (i). □

We now specialize the above result to the nonnegative orthant in R^n. In this setting, the normal map reduces to the Minty map. Also, the orthants of R^n form a polyhedral subdivision corresponding to the Minty map. Moreover, the determinants of the matrices corresponding to the Minty map are precisely the principal minors of the matrix M.

Corollary 1 (Murthy, Parthasarathy and Sabatini [14]). *For the linear complementarity problem corresponding to the matrix M, the following are equivalent*:
(i) *For all $q \in R^n$, $\mathscr{S}(q) \neq \emptyset$ and the mapping $q \mapsto \mathscr{S}(q)$ is Lipschitzian on R^n.*
(ii) *For all $q \in R^n$, LCP(M, q) has a unique solution.*

We add a few remarks regarding the surjectivity condition in part (i) of the above theorem. By assuming the Lipschitzian property only for those vectors q with $\mathscr{S}(q) \neq \emptyset$, we can infer from Theorem 7 that the matrices corresponding to the Minty map are nonsingular, i.e., in the LCP terminology, M is a nondegenerate matrix. In [22], Stone shows that in addition to this nondegenerateness, the matrix belongs to the INS class (which means that in the interior of the set of all solvable qs, the cardinality of the solution set is a constant). He also proves the converse under a so-called Lipschitz path-connectedness assumption.

We now consider an application of the global error bound result (Theorem 8) to affine variational inequalities. Let f be the normal map given by (3). We fix $q \in R^n$ and suppose that for some positive number α,

$$\operatorname{dist}(z, f^{-1}(-q)) \leqslant \alpha \, \| f(z) + q \| \quad \text{for all } z \in R^n. \tag{9}$$

Since $\Pi_{\mathscr{X}}$ is nonexpansive and $\mathscr{S}(q) = \Pi_{\mathscr{X}}(f^{-1}(-q))$, we have with $\beta = \alpha$,

$$\operatorname{dist}(\Pi_{\mathscr{X}}(z), \mathscr{S}(q)) \leqslant \beta \, \| f(z) + q \| \quad \text{for all } z \in R^n, \tag{10}$$

i.e., $\| f(z) + q \|$ can be used as a measure of how far $\Pi_{\mathscr{X}}(z)$ is from the solution set $\mathscr{S}(q)$. Now suppose that (10) holds. With $z = \Pi_{\mathscr{X}}(z) + z - \Pi_{\mathscr{X}}(z)$ and (7), simple algebraic manipulations lead to (9) with $\alpha = 1 + \beta + \beta \| M \|$, where $\| M \|$ denotes the (operator) norm of M. Since the normal map f is open (equivalently, coherently oriented) if and only if it is a homeomorphism, Theorem 8 implies the following result.

Theorem 11. *Let M, $\mathscr{K}$ and f be as described above. Then $\mathscr{S}(q) \neq \emptyset$ for all q and there exists a positive number δ such that*

$$\operatorname{dist}(\Pi_{\mathscr{K}}(z), \mathscr{S}(q)) \leqslant \delta \| f(z) + q \| \quad \text{for all } z \in R^n \text{ and } q \in R^n \tag{11}$$

if and only if $\mathscr{S}(q)$ is a singleton for all $q \in R^n$.

In the context of the LCP, the above theorem says that the global error bound

$$\operatorname{dist}(z^+, \mathscr{S}(q)) \leqslant \delta \| Mz^+ - z^- + q \| \quad \text{for all } z \in R^n \text{ and } q \in R^n \tag{12}$$

holds for a **Q**-matrix M if and only if M is a $\boldsymbol{P}$-matrix. In [7] such a result was proved for the global error bound

$$\operatorname{dist}(z, \mathscr{S}(q)) \leqslant \delta \| z \wedge (Mz + q) \| \quad \text{for all } z \in R^n \text{ and } q \in R^n, \tag{13}$$

where "$\wedge$" denotes the componentwise minimum.

6. Concluding remarks

In this paper, we established a connection between the openness property of a polyhedral multifunction and the Lipschitzian property of its inverse. As a consequence, we showed that a piecewise affine function (from R^n into itself) is surjective and its inverse is Lipschitzian if and only if the function is coherently oriented. It is possible to study the local versions of these results. Such local versions and their connection to openness, metric regularity, and pseudo-Lipschitzian properties are studied in [8]. See also a recent paper of Dontchev and Rockafellar [3] where lower semicontinuity, strong regularity, and the pseudo-Lipschitzian properties are studied in the context of (affine) variational inequalities.

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