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More on the Lost Cousin of the Fundamental Theorem of Algebra

ROMAN SZNAJDER
Bowie State University
Bowie, MD 20715-9465
rsznajder@bowiestate.edu

In his recent note [**2**], Timo Tossavainen proves what he calls "The Lost Cousin of the Fundamental Theorem of Algebra," which we state as:

EXPONENTIAL THEOREM. *For any integer $n \geq 1$, let $0 < \kappa_0 < \kappa_1 < \cdots < \kappa_n$ and a_j (for $j = 0, \ldots, n$) be real numbers with $a_n \neq 0$. Then the function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(t) = \sum_{j=0}^{n} a_j \kappa_j^t$$

has at most n positive *zeros.*

Years ago, I was presented by a friend with a copy of a concise monograph [**1**] (112 pages long) on selected topics in polynomial approximation. In this book, apparently unknown to western readers, the following fact and its proof appear:

GENERALIZED POLYNOMIAL THEOREM. *A function g given by the formula*

$$g(x) = a_0 x^{\alpha_0} + a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n},$$

where $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ are arbitrary real numbers and $a_n \neq 0$, has no more than n roots.

Proof. We proceed by induction on n, noting that for $n = 1$ the statement is obvious. Assume that for some n the claim is true, but for $n + 1$, it is not. Hence, for some real numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1}$ and $a_{n+1} \neq 0$, there is a function

$$g(x) = a_0 x^{\alpha_0} + a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n} + a_{n+1} x^{\alpha_{n+1}},$$

whose number of positive roots is larger than $n + 1$. These roots are identical with the roots of the new function

$$g(x)/x^{\alpha_0} = a_0 + a_1 x^{\alpha_1 - \alpha_0} + \cdots + a_n x^{\alpha_n - \alpha_0} + a_{n+1} x^{\alpha_{n+1} - \alpha_0}.$$

By Rolle's theorem, the derivative of the above function, which has the form

$$b_0 x^{\beta_0} + b_1 x^{\beta_1} + \cdots + b_n x^{\beta_n},$$

with $\beta_0 < \beta_1 < \cdots < \beta_n$ and $b_n \neq 0$, has more than n roots. This contradiction to the induction hypothesis concludes the proof. ■

The Exponential Theorem generalizes the fundamental theorem of algebra to exponential functions the way the Generalized Polynomial Theorem does for generalized polynomials. A striking fact is that the two proofs follow the same path. Despite appearances, the theorems are equivalent, as the following argument shows.

Let $f(t) = \sum_{j=0}^{n} a_j \kappa_j^t$ with $0 < \kappa_0 < \kappa_1 < \cdots < \kappa_n$, $a_j \in \mathbb{R}$, and $a_n \neq 0$. Let $\kappa_0 = e^{c_0}, \kappa_1 = e^{c_1}, \ldots, \kappa_n = e^{c_n}$ for some $c_0 < c_1 < \cdots < c_n$. By multiplying $f(t)$ by Δ^t for a suitable $\Delta > 1$, we may assume that $c_0 > 0$ to ascertain that $1 < c_1/c_0 < \cdots < c_n/c_0$. Then

$$\begin{aligned} f(t) &= a_0 e^{c_0 t} + a_1 e^{c_1 t} + \cdots + a_n e^{c_n t} \\ &= a_0 e^{c_0 t} + a_1 \left(e^{c_0 t}\right)^{c_1/c_0} + \cdots + a_n \left(e^{c_0 t}\right)^{c_n/c_0} \\ &= a_0 x + a_1 x^{c_1/c_0} + \cdots + a_n x^{c_n/c_0} = g(x) \end{aligned}$$

with $x = e^{c_0 t}$. By the Generalized Polynomial Theorem, with $\alpha_0 = 1, \alpha_1 = c_1/c_0, \ldots, \alpha_n = c_n/c_0$, there exist at most k positive roots of the corresponding function $g(x)$. Certainly, when x_i is such a root, $t_i := (\ln x_i)/c_0$ becomes a root of $f(t)$, and vice versa. This way, we have shown that the Generalized Polynomial Theorem implies the Exponential Theorem. The opposite implication comes from reversing the argument.

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Closed Knight's Tours with Minimal Square Removal for All Rectangular Boards

JOE DEMAIO
Kennesaw State University
Kennesaw, GA 30144
jdemaio@kennesaw.edu

THOMAS HIPPCHEN
Kennesaw State University
Kennesaw, GA 30144
thippchen@gmail.com

Finding a closed knight's tour of a chessboard is a classic problem: Can a knight use legal moves to visit every square on the board and return to its starting position? [**1, 3**] An open knight's tour is a knight's tour of every square that does not return to its starting position. While originally studied for the standard 8×8 board, the problem is easily generalized to other rectangular boards. In 1991 Schwenk classified all rectangular boards that admit a closed knight's tour [**2**]. He described every board that cannot admit a closed knight's tour and constructed closed knight's tours for all other boards.