ON THE PSEUDO-LIPSCHITZIAN BEHAVIOR OF THE INVERSE OF A PIECEWISE AFFINE FUNCTION*

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Abstract. In this paper, we investigate the pseudo-Lipschitzian behavior of the inverse of a piecewise affine function around a point. Specifically, we show that for a piecewise affine function $f : R^n \to R^n$, the inverse map f^{-1} is pseudo-Lipschitzian (equivalently, lower semicontinuous) around a point (y^*, x^*) with $y^* \in int(ran\ f)$ if and only if the (essentially active) matrices of f at x^* have the same nonzero determinantal sign. When the branching number of f at x^* is less than or equal to four, this property is equivalent to f being one-to-one in a neighborhood of x^* with Lipschitzian inverse. Using these results, we study the properties of a general piecewise affine function whose inverse is globally Lipschitzian. Our analysis, when specialized to affine variational inequalities and linear complementarity problems, recovers some recently established results of Dontchev and Rockafellar [12] and Stone [41], [42]. Under a strong first order approximation assumption, we show that the inverse of a PC^1 function g is pseudo-Lipschitzian around a point (y^*, x^*) with $y^* \in int(ran\ g)$ if and only if Qi's generalized Jacobian of g at x^* consists of matrices with the same nonzero determinantal sign.

Key words. Piecewise affine function, pseudo-Lipschitzian mapping, Lipschitz path-connected.

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1. Introduction.

Consider a single-valued function $f : R^n \to R^n$ and a point (x^*, y^*) in the graph of f. Following Aubin-Frankowska [4], we say that f^{-1} (the multivalued inverse of f) is *pseudo-Lipschitzian* around (y^*, x^*) if there exist a positive constant α, a neighborhood U^* of x^*, and a neighborhood V^* of y^* such that

$$(1) \qquad f^{-1}(y_1) \cap U^* \subseteq f^{-1}(y_2) + \alpha ||y_1 - y_2|| B \quad \text{for all} \quad y_1, y_2 \in V^* \cap ran\, f$$

where *ran* f denotes the range of f and B denotes the closed unit ball in R^n; if $U^* = R^n$, we say that f^{-1} is *Lipschitzian* around y^*. Clearly, the above definitions can be formulated for any multivalued mapping F (instead of f^{-1}, see Definition 3 below) in a very general setting (e.g., that of a metric space). The pseudo-Lipschitzian property (1) with $V^* \subseteq ran\, f$ is known to be equivalent to f^{-1} being open at linear rate, and to f^{-1} being metrically regular, see Borwein and Zhuang [5] and Penot [27]. Other important characterizations are due to Rockafellar [35] and Mordukhovich [23]. Extensions of the Banach open mapping theorem to nonlinear functions (described by theorems of Lyusternik [22] and Graves [16]) and to multivalued mappings with closed convex graphs (described by theorems of Robinson [31], [32], and Ursescu [44]) are stated in terms of the pseudo-Lipschitzian property. Accounts of these can be found in the book by Aubin and Frankowska [4] and in the papers [9], [10], [11].

In this paper, we are interested in the pseudo-Lipschitzian behavior of the inverse of a piecewise affine function. Unlike the previously mentioned characterizations (which can be viewed as topological and/or analytical), our description is in terms of numerical quantities and somewhat algebraic in nature. We show in this paper that the pseudo-Lipschitzian property of the inverse of a piecewise affine function f around

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a point (y^*, x^*) with $y^* \in int(ran\ f)$ is equivalent to f being coherently oriented at x^*, i.e., the determinants of the (essentially active) matrices that describe f at x^* have the same nonzero sign. We show further that this property reduces to f being locally one-to-one with Lipschitzian inverse when the branching number of f at x^* is less than or equal to four. Under a strong first order approximation, we show that the inverse of a PC^1 function g is pseudo-Lipschitzian around a point (y^*, x^*) if and only if the Qi's generalized Jacobian of g at x^* consists of matrices with the same nonzero determinantal sign. The motivation for such results comes from three recent developments.

In [15], Gowda and Sznajder show that for a piecewise affine function $f : R^n \to R^n$, f is surjective and f^{-1} is Lipschitzian on (all of) R^n if and only if f is open (equivalently, f is coherently oriented). In addition, they show that these equivalent conditions imply that f is one-to-one when the branching number of f is less than or equal to four. The pseudo-Lipschitzian results for piecewise affine functions mentioned in the previous paragraph were predicted in [15] and can be considered as local versions of these (global) results.

When f is the normal map associated with the affine variational inequality AVI$(M, \mathcal{K}, q)$, i.e.,

$$f(x) = M\Pi_{\mathcal{K}}(x) + x - \Pi_{\mathcal{K}}(x) \tag{2}$$

where $M \in R^{n\times n}$ and $\mathcal{K}$ is a (convex) polyhedral set in R^n, and $\Pi_{\mathcal{K}}(x)$ is the projection of x onto $\mathcal{K}$, Dontchev and Rockafellar [12] have shown that f^{-1} is pseudo-Lipschitzian around a point (q^*, x^*) with $q^* = f(x^*) \in int(ranf)$ if and only if f is one-to-one in a neighborhood of x^* with a Lipschitzian inverse. They prove this by employing machinery specific to affine variational inequalities, in particular, by using the so called critical cone. Our result, Theorem 3.2, can be considered as a generalization of Dontchev-Rockafellar result since it is known that for the normal map, the branching number is four [38].

In [26], Pang and Ralph have shown that a PC^1 function f is locally one-to-one with a Lipschitzian inverse around a point x^* if and only if the matrices in Qi's generalized Jacobian of f at x^* have the same nonzero determinantal sign and the (topological) index of f at x^* is ±1. Our Theorem 3.5 can be considered as a pseudo-Lipschitzian analog of this result.

This paper also studies the Lipschitzian behavior of the inverse of a piecewise affine function on subsets of the range of the function. Our motivation for this study comes from recent results of Stone [41], [42]. In connection with the linear complementarity problem (which is AVI(M, R^n_+, q)) and the corresponding normal map $f(x) = Mx^+ - x^-$, Stone [41] (see also [39]) has shown that when f^{-1} is Lipschitzian on the range of f, the matrix M is necessarily nondegenerate and $|f^{-1}(q)|$ is finite and independent of q in the interior of the range of f. Stone [42] proves that these are also sufficient when the range has the so called Lipschitz path-connectedness property. In section 4, we partially extend Stone's results to piecewise affine functions with branching number less than or equal to four.

2. Preliminaries. A *piecewise affine function* $f : R^n \to R^n$ is a continuous function with *domain* R^n for which there exists a set of triples (Ω_j, A_j, a_j) $(j = 1, 2, \ldots, K)$ such that each Ω_j is a (convex) polyhedral set in R^n with nonempty interior, $A_j \in R^{n\times n}$, $a_j \in R^n$, and

(a) $R^n = \cup_{j=1}^K \Omega_j$;
(b) For $i \neq j$, $\Omega_i \cap \Omega_j$ is either empty or a proper common face of Ω_i and Ω_j. In particular, $\mathit{int}\,\Omega_j \cap \mathit{int}\,\Omega_i = \emptyset$ for $i \neq j$;
(c) $f(x) = A_j x + a_j$ on Ω_j, $j = 1, 2, \dots, K$.

The collection $\{\Omega_j : j = 1, 2, \dots, K\}$ is called the polyhedral subdivision of R^n corresponding to f. We shall say that f is *coherently oriented* if all the (square) matrices A_j have the same nonzero determinantal sign. The *branching number* of the polyhedral subdivision $\{\Omega_j : j = 1, 2, \dots, K\}$ of f, or simply that of f, is the maximal number of these Ωs having a common face of dimension $n - 2$. For a comprehensive treatment of piecewise affine functions see [13] or [37]. In the latter reference a piecewise affine function is defined, equivalently, as a single-valued continuous function $f : R^n \to R^n$ for which there exist finitely many affine functions $f_j : R^n \to R^n$ such that for all $x \in R^n$,

$$f(x) \in \{f_1(x), f_2(x), \dots, f_J(x)\}. \tag{3}$$

More generally, a function g from a subset $\mathcal{D}$ of R^n into R^n is a PC^1 function on $\mathcal{D}$ if g is continuous and there exist continuously differentiable functions $g_j : \mathcal{D} \to R^n$ such that

$$g(x) \in \{g_1(x), g_2(x), \dots, g_J(x)\} \quad \text{for all} \quad x \in \mathcal{D};$$

If each g_i is affine, we shall say that g is *piecewise affine on* $\mathcal{D}$. For basic properties of PC^1 functions, see [37]. It is known that a PC^1 function g on an open set $\mathcal{D}$ is B-differentiable at every point $x^* \in \mathcal{D}$, that is,

$$\lim_{x \to x^*} \frac{g(x) - [g(x^*) + g'(x^*; x - x^*)]}{\|x - x^*\|} = 0$$

where

$$g'(x^*; x) := \lim_{t \downarrow 0} \frac{g(x^* + tx) - g(x^*)}{t}$$

exists for each x. Moreover, the B-derivative $g'(x^*; \cdot)$ is a piecewise linear function given by

$$g'(x^*; x) \in \{\nabla g_i(x^*) x : i \in I^e(g, x^*)\}$$

where $I^e(g, x^*)$ is the collection of (so called essentially active) indexes i such that $x^* \in \mathit{cl}\ \mathit{int}\, \{z : g(z) = g_i(z)\}$ (Prop. 4.1.3, [37]). The set

$$\partial_Q g(x^*) := \{\nabla g_i(x^*) : i \in I^e(g, x^*)\}$$

is called Qi's *generalized Jacobian* [28]. It is worth noting that the convex hull of this set is the generalized Jacobian $\partial g(x^*)$ [7]. Also, for a piecewise affine function f which is described by the triples (Ω_j, A_j, a_j),

$$\partial_Q f(x^*) := \{A_j : x^* \in \Omega_j\}.$$

We shall say that g (or f) is *coherently oriented* at x^* if all matrices in $\partial_Q g(x^*)$ (respectively, $\partial_Q f(x^*)$) have the same nonzero determinantal sign. We define the *branching number* of f at x^* as the branching number of the polyhedral subdivision induced by the piecewise affine function $f'(x^*;\,\cdot\,)$. We note that a piecewise affine function f is coherently oriented at x^* if and only if the corresponding piecewise linear mapping $f'(x^*;\,\cdot\,)$ is coherently oriented on R^n.

We now recall various (Lipschitz) continuity concepts, see [4] for full details. In the definitions below, F denotes a multivalued mapping from R^n into itself with domain and range denoted by *dom* F and *ran* F, respectively. We let $G = F^{-1}$ and write $\mathrm{Gr}\,F$ for the graph of F; we fix $(x^*, y^*) \in \mathrm{Gr}\,F$ so that $(y^*, x^*) \in \mathrm{Gr}\,G$. For a set U, we write $F(U) := \{y : y \in F(x) \text{ for some } x \in U\}$.

DEFINITION 1. *We say that* G *is lower semicontinuous at a point* $(\bar{y}, \bar{x}) \in \mathrm{Gr}\,G$, *if for every sequence* $\{y^m\}$ *in* $dom\,G$ *converging to* $\bar{y}$, *there exists a sequence* $\{x^m\}$ *in* $ran\,G$ *converging to* $\bar{x}$ *such that* $(y^m, x^m) \in \mathrm{Gr}\,G$. *We shall say that* G *is lower semicontinuous around a point* $(y^*, x^*) \in \mathrm{Gr}\,G$ *if there exists a neighborhood* $V^* \times U^*$ *of* (y^*, x^*) *in* $R^n \times R^n$ *such that* G *is lower semicontinuous at each point* $(\bar{y}, \bar{x}) \in (V^* \times U^*) \cap \mathrm{Gr}\,G$. *By abuse of language, we shall say that* G *is lower semicontinuous at a point* y^* *if* G *is lower semicontinuous at any point* (y^*, u^*) *in the graph of* G. *Finally,* G *is said to be lower semicontinuous on a set* $E \subset dom\,G$ *if* G *is lower semicontinuous at each point of* E.

DEFINITION 2. *We say that* F *is open around* x^* *if there exists an open set* U^* *containing* x^* *such that for any open set* $U \subseteq U^*$, $F(U)$ *is open in* R^n.

The following simple result connects the above two concepts; the conclusion of this result can be obtained without any assumptions on F by slightly changing the definition of lower semicontinuity, see Proposition 1.4.4 in [4].

PROPOSITION 2.1. *Suppose that* F *is single-valued and continuous at* x^*. *Then* F^{-1} *is lower semicontinuous around* (y^*, x^*) *with* $y^* = F(x^*) \in int(ran\,F)$ *if and only* F *is open around* x^*.

Proof. Suppose that F^{-1} is lower semicontinuous around (y^*, x^*) and $y^* \in int(ran\,F)$; let U^* and V^* be as in Definition 1. We may assume that $V^* \subseteq ran\,F$ and (by the continuity of F) that $F(U^*) \subseteq V^*$. Let U be an arbitrary open set contained in U^* and assume, if possible, that $F(U)$ is not open in R^n. Then there exist a point $\bar{y} \in F(U)$ and a sequence $\{y^m\}$ such that y^m converges to $\bar{y}$ and $y^m \notin F(U)$ for all m. Let $\bar{y} = F(\bar{x})$ for some $\bar{x} \in U$. By the lower semicontinuity property, there exists a sequence $\{x^m\}$ such that $x^m \to \bar{x}$ and $y^m = F(x^m)$ for all m. But then $x^m \in U$ for large m implying $y^m \in F(U)$ for all such m. This contradiction proves the openness of $F(U)$ and that of F around x^*.

Now suppose that F is open around x^*. Let U^* be as in Definition 2, and $V^* := F(U^*)$. Consider any $(\bar{y}, \bar{x}) \in (V^* \times U^*) \cap \mathrm{Gr}\,G$ and a sequence $\{y^m\}$ in $ran\,F$ converging to $\bar{y}$. For each natural number l, $\bar{y}$ is an interior point of $F(\bar{x} + \frac{1}{l}B)$ and hence for an appropriate m_l we have $m \geq m_l \Longrightarrow y^m \in F(\bar{x} + \frac{1}{l}B)$. Then there exists $x_l^m \in \bar{x} + \frac{1}{l}B$ such that $y^m = F(x_l^m)$ for all $m \geq m_l$. We may assume that the sequence $\{m_l\}$ is strictly increasing. Then the sequence $\{x_1^{m_1}, \ldots, x_1^{m_2-1}, x_2^{m_2}, \ldots, x_2^{m_3-1}, x_3^{m_3}, \ldots\}$ clearly converges to $\bar{x}$ and $y^m = F(x^m)$ for all $m \geq m_1$. Thus G is lower semicontinuous at $(\bar{y}, \bar{x})$, and hence around $(F(x^*), x^*)$. $\square$

DEFINITION 3. *We say that*

(a) G *is* pseudo-Lipschitzian *around* $(y^*, x^*) \in \mathrm{Gr}\,G$ *if there exist a positive constant* α, *a neighborhood* U^* *of* x^* *and a neighborhood* V^* *of* y^* *such that*

$$G(y_1) \cap U^* \subseteq G(y_2) + \alpha \|y_1 - y_2\| B \quad \textit{for all} \quad y_1, y_2 \in V^* \cap dom\,G;$$

(b) G is locally lower Lipschitzian *at $y^* \in dom\, G$ if there exist a positive number β and a neighborhood V^* of y^* such that*

$$G(y^*) \subseteq G(y) + \beta \|y - y^*\| B \quad \text{for all} \quad y \in V^* \cap dom\, G;$$

(c) G is Lipschitzian *on a set $\mathcal{D} \subset dom\, G$ if there exists a positive number λ such that*

$$G(y) \subseteq G(z) + \lambda \|y - z\| B \quad \text{for all} \quad y, z \in \mathcal{D}.$$

It should be noted that the original definition of pseudo-Lipschitzian property given by Aubin [3] and used in the papers of Dontchev [9] and Dontchev and Rockafellar [12] demands, in addition to what is given above, that $y^* \in int(dom\, G)$.

It is easily seen that if G is locally lower Lipschitzian at $y^* \in dom\, G$, then G is lower semicontinuous at y^*.

The following result will be used in Section 4. It generalizes a result of Stone (Theorem 4.4, [40]) proved in the setting of linear complementarity problem for the mapping $f(x) = Mx^+ - x^-$.

PROPOSITION 2.2. *Consider a piecewise affine function $f : R^n \to R^n$ described by the triples (Ω_j, A_j, a_j) $(j = 1, 2, \ldots, K)$. If each A_j $(j = 1, \ldots, K)$ is nonsingular, then $int(ran\, f)$ is nonempty, open, and connected.*

Proof. Assume that every A_j is nonsingular. Clearly, for each j, $D_j := A_j(int\, \Omega_j)) + a_j$ is open (and nonempty) and so $int(ran\, f)$ is open and nonempty. We now prove the connectedness of $int(ran\, f)$. Assume, if possible, that $int(ran\, f)$ is not connected and consider its (disjoint) components. Then for each j, D_j, being a connected set, must belong to some component of $int(ran\, f)$. Let D be a component of $int(ran\, f)$ containing the maximal number, say, L of D_js. Without loss of generality, let $\cup_{j=1}^{L} D_j \subset D$. Since $int(ran\, f)$ contains an (open) component different from D, it follows that $L < K$. Now $\cup_1^L \Omega_j$ is closed, has nonempty interior and not equal to R^n; hence has an $(n-1)$-dimensional boundary (Cf. [17], Cor. IV.4.2). There is a boundary point x^* of this set which lies in the relative interior of an $(n-1)$-dimensional face Z of, say, Ω_1. This x^* must belong to some other Ω_i with $i \neq 1$. Since $\Omega_1 \cap \Omega_i$ is a common proper face of Ω_1 and Ω_i, it follows that Z is a face of Ω_i also. The index i must be greater than L else a neighborhood of x^* would be contained in $int(\Omega_1 \cup \Omega_i)$ contradicting the choice of the boundary point x^*. Without loss of generality let $i = L + 1$. We now show that any point p in D_{L+1} can be connected to any point q in D_1 by a path in $int(ran\, f)$ proving that D must contain D_{L+1} leading to a contradiction to the choice of L. Fix $p \in D_1$ and $q \in D_{L+1}$. First observe that $f(Z)$ is $(n-1)$-dimensional and $f(x^*)$ is the relative interior of $f(Z)$. If the matrices A_1 and A_{L+1} have the same determinantal sign, then p and q are on "opposite sides" of $f(Z)$ and $f(x^*)$ is in the interior of $f(\Omega_1) \cup f(\Omega_{L+1})$. In this case, the line segment $[p, f(x^*)]$ joined by the line segment $[f(x^*), q]$ defines a path in $int(ran\, f)$. If the matrices A_1 and A_{L+1} have opposite determinantal signs, then p and q are on the "same side" of $f(Z)$. In this case, $D_1 \cap D_{L+1}$ will have a common point, say, y^*. We can then consider the path in $int(ran\, f)$ defined by the line segments $[p, y^*]$ and $[y^*, q]$. Thus in either case, p and q can be connected by a path that lies in $int(ran\, f)$. (This somewhat geometric proof can be made precise and simple by assuming, without loss of generality, $x^* = 0$, $f(x^*) = 0$, $Z \subseteq R^{n-1} \times \{0\}$, $A_1 = I$ (the identity matrix). In this setting, the columns of A_{L+1} are $e_1, e_2, \ldots, e_{n-1}, d$ where e_i is the ith coordinate vector in R^n and d is some vector in R^n.) □

3. Pseudo-Lipschitzian properties. We begin with a result which is obtained by slightly modifying and combining Theorems 5 and 6 in [15].

THEOREM 3.1. *For a piecewise affine function f from R^n into itself, the following are equivalent:*

(*i*) *f is surjective and f^{-1} lower semicontinuous at all points (y^*, x^*) with $y^* = f(x^*)$.*

(*ii*) *f is an open map from R^n into R^n, i.e., f maps open sets to open sets.*

(*iii*) *f is coherently oriented.*

(*iv*) *f is surjective and f^{-1} is Lipschitzian on R^n.*

(*v*) *f is surjective and f^{-1} is locally lower Lipschitzian at each point of R^n.*

Furthermore, when the branching number of f is less than or equal to four, each of the above conditions is equivalent to

(*vi*) *f is one-to-one, or equivalently, f is a homeomorphism.*

Remarks. Conditions (*ii*) – (*vi*) appear in [15]. The equivalence of (*i*) and (*ii*) follows from Proposition 2.1. The motivation for adding the lower semicontinuity condition (*i*) comes from [12], see the remarks following Theorem 3.2.

Our main result of this section is the local version of the above theorem.

THEOREM 3.2. *For an $f \in \mathcal{PA}(R^n, R^n)$, the following statements are equivalent:*

(*a*) *f^{-1} is lower semicontinuous around (y^*, x^*) with $y^* = f(x^*) \in int(ran\, f)$;*

(*b*) *f is open around x^*;*

(*c*) *f is coherently oriented at x^*;*

(*d*) *f^{-1} is pseudo-Lipschitzian around (y^*, x^*) with $y^* = f(x^*) \in int(ran\, f)$.*

Moreover, when either (i) the branching number of f at x^* is less than or equal to four, or

(ii) index $(f, x^*) = \pm 1$, each of the above statements is equivalent to

(*e*) *f is one-to-one around x^* and f^{-1} is Lipschitzian around y^*.*

Proof. By considering $f(x + x^*) - f(x^*)$, if necessary, we may assume that $x^* = y^* = 0$. The implication $(a) \Longrightarrow (b)$ follows from Proposition 2.1. To see $(b) \Longrightarrow (c)$, consider, $\widehat{f}(x) := f'(0; x)$. In a neighborhood of the origin, $\widehat{f}$ and f coincide. When (*b*) holds, $\widehat{f}$ is open in a neighborhood of the origin; hence open on R^n by positive homogeneity of $\widehat{f}$. It follows that $\widehat{f}$ is coherently oriented [37] proving (*c*). Now suppose that (*c*) holds so that $\widehat{f}$ is coherently oriented on R^n. Let U^* be a neighborhood of the origin on which $\widehat{f}$ and f coincide. Since the matrices describing $\widehat{f}$ are finite in number and nonsingular, it follows that for each $y \in R^n$, the set $(\widehat{f})^{-1}(y)$ is finite; moreover, we can find a neighborhood V^* of the origin such that $(\widehat{f})^{-1}(y) \subseteq U^*$ for all $y \in V^*$. It follows from the previous theorem, that $(\widehat{f})^{-1}$ is Lipschitzian on R^n. Hence there exists a positive constant α such that

$$(\widehat{f})^{-1}(y_1) \subseteq (\widehat{f})^{-1}(y_2) + \alpha \|y_1 - y_2\| B \tag{4}$$

for all y_1 and y_2. Restricting y_1, y_2 in V^*, we see, for $i = 1, 2$, that $f^{-1}(y_i) \cap U^* = (\widehat{f})^{-1}(y_i) \cap U^* = (\widehat{f})^{-1}(y_i)$. Hence from the above inclusion we have

$$f^{-1}(y_1) \cap U^* \subseteq f^{-1}(y_2) + \alpha \|y_1 - y_2\| B \quad \text{for all} \quad y_1, y_2 \in V^* \tag{5}$$

proving (*d*). The implication $(d) \Longrightarrow (a)$ follows easily from the definitions.

When condition (*i*) holds, the branching number of $\widehat{f}$ is less than or equal to four.

It follows from a result of Kuhn and Löwen [20] that $\widehat{f}$ is a homeomorphism. $(\widehat{f})^{-1}$, being piecewise linear, is Lipschitzian on R^n. Since $f = \widehat{f}$ near the origin, it follows that f is one-to-one around the origin with a Lipschitzian inverse.

When condition *(ii)* holds, the stated conclusion follows from Theorem 4, [26]; it can also be deduced from Theorem 5, [14] (applied to $\widehat{f}$). □

Remarks. To discuss a particular and important case of Theorem 3.2, consider a matrix $M \in R^{n\times n}$ and a (convex) polyhedral set $\mathcal{K} \subseteq R^n$. Correspondingly, we define the normal map f by (2). Given a vector $q \in R^n$, the solvability of the equation $f(x)+q = 0$ is equivalent to the solvability of the affine variational inequality problem AVI$(M, \mathcal{K}, q)$: find $u^* \in \mathcal{K}$ such that

$$\langle Mu^* + q, u - u^*\rangle \geq 0 \quad \text{for all} \quad u \in \mathcal{K}. \tag{6}$$

Robinson [34] made a formal study of the normal map and showed that it is piecewise affine, and coherently oriented if and only if it is a homeomorphism. Ralph [29] proved that branching number of the normal map is less than or equal to four and recently Scholtes [38] has shown that the branching number is four. In this setting, Dontchev and Rockafellar have shown, while studying the strong regularity of the AVI at a solution point, the equivalence of statements (a), (d) and (e) in Theorem 3.2 for the normal map. Their proof of the implication $(a) \Longrightarrow (e)$ consists in applying the above mentioned result of Robinson to the normal map corresponding to the "critical cone" at the solution point under consideration. (Note that the implications $(e) \Longrightarrow (d) \Longrightarrow (a)$ are immediate.)

The following corollary is obvious.

COROLLARY 3.3. *Suppose $f \in \mathcal{PA}(R^n, R^n)$ and int(ran f) $\neq \emptyset$. Let $\mathcal{D}$ be a nonempty open subset of ran f. If f^{-1} is lower semicontinuous on $\mathcal{D}$, then every matrix in the set*

$$\bigcup_{\bar{x}\in f^{-1}(\mathcal{D})} \partial_Q f(\bar{x})$$

is nonsingular.

This corollary is somewhat similar to the following result which is a modified form of Theorem 7 in [15].

PROPOSITION 3.4. *Suppose $f \in \mathcal{PA}(R^n, R^n)$ and int(ran f) $\neq \emptyset$. If f^{-1} is lower semicontinuous (or Lipschitzian) on ran f, then all the matrices of f are nonsingular.*

We end this section with a generalization of Theorem 3.2 to PC^1 functions. Consider a PC^1 function g (see Section 2) defined on an open set $\mathcal{D}$ containing the point x^*; let $g'(x^*;\cdot)$ denote the B-derivative of g at x^* and let $h(x) := g(x^*)+g'(x^*; x-x^*)$. We say that g is *strongly B-differentiable* if

$$\lim_{(u,v)\to(x^*,x^*)} \frac{g(u) - g(v) - [h(u) - h(v)]}{||u - v||} = 0.$$

By specializing the (Inverse Mapping) Theorem given in [10] to PC^1 functions, we get the equivalence of the following statements:

(a) g^{-1} is pseudo-Lipschitzian around (y^*, x^*) with $y^* = g(x^*) \in int(ran\, g)$.
(b) h^{-1} is pseudo-Lipschitzian around (y^*, x^*) with $y^* = h(x^*) \in int(ran\, h)$.

Combining the above equivalence and Theorem 3.2, we arrive at the following.

THEOREM 3.5. *Let g be a* PC^1 *function from an open set $\mathcal{D} \subset R^n$ into R^n. Let g be strongly B-differentiable at $x^* \in \mathcal{D}$. Then the following are equivalent:*

(i) g^{-1} is pseudo-Lipschitzian around (y^*, x^*) with $y^* = g(x^*) \in int(ran\, g)$.

(ii) h is coherently oriented at x^*, i.e., the matrices in $\partial_Q g(x^*)$ have the same nonzero determinantal sign.

Remarks. In a private communication, S. Scholtes notes that the above equivalence continues to hold for a PC^1 function g if we assume, instead of the strong B-differentiability of g at x^*, that g is PC^1-equivalent to a PC^1 function which is strongly B-differentiable at x^* (i.e., there exists a local PC^1-homeomorphism ϕ such that $\phi(x^*) = x^*$ and $g \circ \phi$ is strongly B-differentiable at x^*). We refer to [18], [19] for various properties of this PC^1-equivalence and their applications.

4. The Lipschitzian behavior of the inverse. In this section, we address the issue of when the inverse of a piecewise affine function is Lipschitzian on the range (more generally, on an open connected subset of the range) of the function. This study is motivated by some recent results of Stone [41], [42] (see the Introduction) and those in [24], [25], [39] and [15]. Stone's analysis is geometric and specially geared towards the LCP. In this section, we extend Stone's analysis to an arbitrary piecewise affine function whose branching number is less than or equal to four.

We first state the necessary conditions.

THEOREM 4.1. *Suppose that $f \in \mathcal{PA}(R^n, R^n)$, $int(ran\, f) \neq \emptyset$, and branching number of f is less than or equal to four. If f^{-1} is Lipschitzian or lower semicontinuous on $ran\, f$, then the following hold:*

(a) All the matrices corresponding to f are nonsingular.

(b) $int(ran\, f)$ is connected.

(c) There is a natural number k such that $|f^{-1}(q)| = k$ for all $q \in int(ran\, f)$.

(d) When $int(ran\, f)$ is simply connected, there exist functions $g_i : int(ran\, f) \to R^n$ $(i = 1, 2, \ldots, k)$ such that

  (i) Each g_i is piecewise affine and one-to-one;

  (ii) The sets $g_i(int(ran\, f))$ $(i = 1, 2, \ldots, k)$ are disjoint, and

  (iii) $f^{-1}(q) = \{g_1(q), g_2(q), \ldots, g_k(q)\}$ for all $q \in int(ran\, f)$.

The proof of this theorem is based on the following propositions.

PROPOSITION 4.2. *Suppose that $f \in \mathcal{PA}(R^n, R^n)$, $int(ran\, f) \neq \emptyset$ and branching number of f is less than or equal to four. Let $\mathcal{D}$ be a nonempty open connected subset of $int(ran\, f)$. If f^{-1} is lower semicontinuous on $\mathcal{D}$, then there is a natural number k such that $|f^{-1}(q)| = k$ for all $q \in \mathcal{D}$.*

Proof. We assume the description of f given at the beginning of Section 2. For any $q^* \in \mathcal{D}$ and $x^* \in f^{-1}(q^*)$, there exists a matrix-vector pair (A_j, a_j) such that $A_j x^* + a_j = q^*$. Since $A_j \in \partial_Q f(x^*)$, it follows from Corollary 3.3 that A_j is nonsingular. Since there are only K such matrix-vector pairs, we see that $|f^{-1}(q^*)| \leq K$. Let

$$k := \max_{q \in \mathcal{D}} |f^{-1}(q)|.$$

Fix a vector $q^* \in \mathcal{D}$ such that $|f^{-1}(q^*)| = k$. We show that for all q in a neighborhood of q^*, $|f^{-1}(q)| = k$. To see this, let $f^{-1}(q^*) = \{x_1^*, x_2^*, \ldots, x_k^*\}$ and observe that for each $i = 1, 2, \ldots, k$, by Theorem 3.2, there exist neighborhoods U_i of x_i^* and V_i of q^* such that $f : U_i \to V_i$ is one-to-one (actually a homeomorphism) and $f^{-1} : V_i \to U_i$ is Lipschitzian. Without loss of generality, we may assume that these U_is are disjoint. Then for each q in $\cap_{i=1}^k V_i$, $f^{-1}(q)$ has at least k inverse images (one in each U_i) and hence (by the definition of k), exactly k inverse images. To see that this property holds for all q in $\mathcal{D}$, we consider the set $\mathcal{D}_1 := \{q \in \mathcal{D} : |f^{-1}(q)| = k\}$. By the above analysis, $\mathcal{D}_1$ is nonempty and open in $\mathcal{D}$. Since $\mathcal{D}$ is assumed to be connected, to show that $\mathcal{D}_1 = \mathcal{D}$, it remains to show that $\mathcal{D}_1$ is closed in $\mathcal{D}$. To this end, let $\bar{q}$ be an element of $\mathcal{D}$ which is on the boundary of $\mathcal{D}_1$; let $f^{-1}(\bar{q}) = \{z_1, z_2, \ldots, z_l\}$ where $l < k$. Choose $\{q^m\} \subseteq \mathcal{D}_1$ such that $q^m \to \bar{q}$ as $m \to \infty$. Then for each index i, by the lower semicontinuity of f^{-1}, there exists a sequence z_i^m such that $z_i^m \in f^{-1}(q^m)$ and $z_i^m \to z_i$ as $m \to \infty$. We may assume without loss of generality that for each i, the sequence $\{z_i^m\}$ is contained in a neighborhood U_i of z_i such that U_is are disjoint and f is one-to-one on every U_i. Now for each q^m sufficiently close to $\bar{q}$, $f^{-1}(\bar{q})$ contains k elements; hence there exists $\zeta^m \in f^{-1}(q^m)$ such that $\zeta^m \notin \cup_{i=1}^{l} U_i$. The sequence $\{\zeta^m\}$ is bounded as $\zeta^m = A_j^{-1}(q^m - a_j)$ for some $j \in \{1, 2, \ldots, K\}$. Without loss of generality, we may assume that $\zeta^m \to \zeta$. Since $f(\zeta^m) = q^m$, we have $f(\zeta) = \bar{q}$. At the same time, $\zeta \notin \cup_{i=1}^{l} U_i$ which is a contradiction since $f^{-1}(\bar{q}) \subset \cup_{i=1}^{l} U_i$. We have thus shown that $\mathcal{D} = \mathcal{D}_1$, i.e., for each $q \in \mathcal{D}$, $|f^{-1}(q)| = k$. $\square$

An alternate proof of the above proposition can be given using covering maps. Recall that a mapping f from a set $X \subseteq R^n$ into a set $Y \subseteq R^n$ is a covering map if each point y^* in Y has a neighborhood V^* such that $f^{-1}(V^*)$ splits into disjoint sets U_i^* such that $f : U_i^* \to V^*$ is a homeomorphism [2]. A covering map is a local homeomorphism. The converse holds (with $|f^{-1}(y)| < \infty$ for all $y \in Y$) under the assumptions that Y is connected and f is a closed map (i.e., f maps closed sets to closed sets) [6]. If f, in addition to being a local homeomorphism, is also proper, (i.e., the inverse image of a compact set in Y is compact in X), then $|f^{-1}(y)|$ is a finite constant on Y, see [1], Corollary 1.4. (This result can be used to prove Proposition 4.2). Additional information can be obtained when X and Y are path-connected and Y is simply connected. In this context it is known that a closed local homeomorphism f from X into Y is necessarily a homeomorphism, see Theorem 2.3.6 in [37]. We shall use this result in the next proposition.

PROPOSITION 4.3. *Let f and $\mathcal{D}$ be as in the previous proposition. If $\mathcal{D}$ is simply connected and f^{-1} is lower semicontinuous on $\mathcal{D}$, then there exist a natural number k and functions g_i $(i = 1, 2, \ldots, k)$ such that*

(i) Each g_i is piecewise affine and one-to-one from $\mathcal{D}$ into R^n,
(ii) The sets $g_i(\mathcal{D})$ $(i = 1, 2, \ldots, k)$ are disjoint, and
(iii) $f^{-1}(q) = \{g_1(q), g_2(q), \ldots, g_k(q)\}$ for all $q \in \mathcal{D}$.

Proof. We let $Y := \mathcal{D}$, $X := f^{-1}(\mathcal{D})$ and X_i $(i \in \Lambda)$ be the connected components of X. Then by Theorem 3.2, f is a local homeomormphism from X_i to Y. Also, from the nonsingularity of the matrices relevant to $f : X_i \to Y$ (see Corollary 3.3), f is proper and hence closed. Since X_i is path-connected and Y is simply connected (by assumption), we see that $f : X_i \to Y$ is a homeomorphism. Let g_i be the inverse of this function. Since $g_i(q) \in \{A_1^{-1}(q - a_1), A_2^{-1}(q - a_2), \ldots, A_K^{-1}(q - a_K)\}$, for all $q \in Y$, we see that g_i is piecewise affine from Y into X_i. Also, the sets $g_i(Y)$ $(= X_i)$ as i varies over the index set Λ are disjoint. It follows that for any $q \in Y$, $f^{-1}(q)$ consists of distinct elements $g_i(q)$ $(i \in \Lambda)$. The previous proposition proves that the cardinality of Λ is k. $\square$

Proof. (of Theorem 4.1.) Under the assumptions of the theorem, statement (a) follows from Proposition 3.4. Proposition 2.2 gives (b). By taking $\mathcal{D} = int(ran\, f)$, statement (c) follows from Proposition 4.2, and statement (d) follows from Proposition 4.3. $\square$

Now to recover the results of Stone, consider for a matrix $M \in R^{n\times n}$, $f(x) = Mx^+ - x^-$. In this setting, the nonnegative orthant is in the range and it is easily seen by looking at all the orthants that the branching number of f is four. When f^{-1} is Lipschitzian on the range of f, statements $(a) - (c)$ of Theorem 4.1 translate into Stone's results: M is nondegenerate, *int*(*ran* f) is connected, and M is an INS matrix.

Remarks. In Theorem 4.1, the simple connectedness condition is imposed on *int*(*ran* f) to get statement (d). Is this condition superfluous? We do not know the answer even in the LCP setting. We may ask whether *int*(*ran* f) is simply connected when all the matrices of f are nonsingular. An example due to Stone [43] shows that the answer is 'no' for a general f; we suspect that the answer is 'yes' in the LCP setting.

We now turn our attention to specifying sufficient conditions for f^{-1} to be Lipschitzian on *ran* f. We have proved in [15] that for a polyhedral multifunction (it is one whose graph is a finite union of polyhedral sets) with convex range, the openness of the multifunction from the domain into the range is equivalent to the inverse multifunction being Lipschitzian. The proof of this result goes through if the range, instead of being convex, is Lipschitz path-connected [42].

DEFINITION 4. *A nonempty set $\mathcal{D} \subseteq R^n$ is said to be Lipschitz path-connected if there is a constant α such that given any two points $a, b \in \mathcal{D}$, there is a polygonal path γ joining a and b in $\mathcal{D}$ such that $L(\gamma; a, b) \leq \alpha||a - b||$ where $L(\gamma; a, b)$ denotes the length of γ from a to b.*

While convex sets are always Lipschitz path-connected, non-convex connected sets need not have this property; a U-shaped unbounded set in the plane will serve as an example. It can be easily shown that every piecewise affine function defined on a Lipschitz path-connected set in R^n is Lipschitzian on that set.

Some modifications of Theorem 3 and Lemma 1 in [15] (see Appendix for a proof) lead to the following result.

THEOREM 4.4. *Let F be a polyhedral multifunction with dom $F \subseteq R^n$ and ran $F \subseteq R^m$. Suppose Y is a nonempty Lipschitz path-connected subset of ran F and $X := F^{-1}(Y) := \{x : F(x) \cap Y \neq \emptyset\}$. Then $(a) \Longrightarrow (b)$ where*

(a) F^{-1} is lower semicontinuous on Y and

(b) F^{-1} is Lipschitzian on Y.

Remark. Although the reverse implication $(b) \Longrightarrow (a)$ in the above theorem may not hold for a general Y, we note that F^{-1} is lower semicontinuous on *ran* F whenever it is Lipschitzian on *ran* F.

The proof of Theorem 4.4 depends on the following technical lemma which may be of independent interest.

LEMMA 4.5. *Let F be a polyhedral multifunction, Y a nonempty subset of ran F, and $X := F^{-1}(Y)$. Suppose that condition (a) of the above theorem holds. Then there exists a positive number β (depending only on F) such that for any two points p^* and q^* in Y with the line segment $[p^*, q^*] \subseteq Y$, we have*

$$F^{-1}(p^*) \subseteq F^{-1}(q^*) + \beta||p^* - q^*||B\,.$$

We are now ready to state our sufficiency result.

THEOREM 4.6. *Suppose for an* $f \in \mathcal{PA}(R^n, R^n)$

(1) the matrices corresponding to f *are nonsingular,*

(2) f^{-1} *is lower semicontinuous on* $int(ran\, f)$*, and*

(3) $int(ran\, f)$ *is Lipschitz path-connected.*

Then f^{-1} *is Lipschitzian on* $ran\, f$.

Remarks. In the previous version of the paper, we obtained the same conclusion under much more restrictive assumptions. In addition to (*1*)-(*3*) above, we assumed that the branching number of f is less than or equal to four, and that $int(ran\, f)$ is simply connected. Thanks to the suggestions of a referee, we now have the above stronger formulation. Note that except for condition (*3*), the other two conditions are necessary for f^{-1} to be Lipschitzian on $ran\, f$.

Proof. (of Theorem 4.6.) We apply Theorem 4.4 to $F = f$ with $Y = int(ran\, f)$, so (*2*) implies that f^{-1} is Lipschitzian on $int(ran\, f)$. We now show that in the presence of (*1*), this gives the Lipschitzian property of f^{-1}.

Let C denote the Lipschitzian constant of f^{-1} on $int(ran\, f)$. By a result of Robinson [33], there exists a positive number λ such that for each $q \in ran\, f$, and for some neighborhood V of q it holds

$$f^{-1}(\bar{q}) \subseteq f^{-1}(q) + \lambda||\bar{q} - q||B$$

for all $\bar{q} \in V$. We now put

$$L := 2C + [\max||A||]C^2 + (C + \lambda)C$$

where $\max||A||$ denotes the maximum of norms of matrices defining f.

Fix p and q in $ran\, f$. We show that

$$f^{-1}(p) \subseteq f^{-1}(q) + L||p - q||B. \tag{7}$$

Let $x \in f^{-1}(p)$, let Ω be one of the polyhedral sets appearing in the definition of f that contains x, and let $f(x) = Ax + a = p$. Pick $\bar{x} \in int\,\Omega$ with $||x - \bar{x}|| \leq C||p - q||$. Note that $\bar{p} := f(\bar{x}) \in int(ran\, f)$ (because of (*1*)) and $||p - \bar{p}|| = ||A(x - \bar{x})|| \leq C||A||\,||p - q||$.

Now, there exists a neighborhood $V \subseteq q + C||p - q||B$ such that

$$f^{-1}(\bar{q}) \subseteq f^{-1}(q) + \lambda||\bar{q} - q||B \quad \text{for all} \quad \bar{q} \in V.$$

Since the matrices of f are nonsingular, we can take a $\bar{q} \in V \cap int(ran\, f)$ so that this $\bar{q}$ satisfies the previous inclusion and $||\bar{q} - q|| \leq C||p - q||$. By assumption,

$$f^{-1}(\bar{p}) \subseteq f^{-1}(\bar{q}) + C||p - q||B$$

and so corresponding to $\bar{x}$ in $f^{-1}(\bar{p})$, there exists a $\bar{y} \in f^{-1}(\bar{q})$ and $y \in f^{-1}(q)$ such that $||\bar{x} - \bar{y}|| \leq C||\bar{p} - \bar{q}||$ and $||\bar{y} - y|| \leq \lambda||\bar{q} - q||$.

We now have

$$\begin{aligned} ||x - y|| &\leq ||x - \bar{x}|| + ||\bar{x} - \bar{y}|| + ||\bar{y} - y|| \\ &\leq C||p - q|| + C||\bar{p} - \bar{q}|| + \lambda||\bar{q} - q|| \\ &\leq C||p - q|| + C\,[\,||\bar{p} - p|| + ||p - q|| + ||q - \bar{q}||\,] + \lambda||\bar{q} - q|| \\ &\leq 2C||p - q|| + ||A||C^2||p - q|| + (C + \lambda)||\bar{q} - q|| \\ &\leq L||p - q|| \end{aligned}$$

proving (7). This completes the proof of the theorem. □

In the context of LCP, based on the mapping $f(x) = Mx^+ - x^-$, Stone [42] has shown that f^{-1} is Lipschitzian on *ran* f if M is nondegenerate, is in the INS class, and *int*(*ran* f) is Lipschitz path-connected. In other words, he obtains the conclusion of the above theorem (for the LCP) by replacing condition (*2*) by

(*2'*) For some natural number k, $|f^{-1}(q)| = k$ for all $q \in$ *int*(*ran* f).

It is not known if the same is true in the general setting. It is also not known if the Lipschitz path-connectedness condition on *int*(*ran* f) can be removed in the LCP setting as well as in the general setting. We end this section by noting a result of Luo and Tseng [21] which says that in the context of LCP, the solution map $q \mapsto \mathrm{SOL}(M,q)$ is Lipschitzian on the set of all solvable qs if and only if for every solvable q, for every $x \in \mathrm{SOL}(M,q)$ and for every nonzero $w \in R^n$ satisfying $\mathrm{SOL}(M, q+\epsilon w) \neq \emptyset$ for $\epsilon > 0$ sufficiently small, it holds that $d(x, \mathrm{SOL}(M, q+\epsilon w)) \to 0$ as $\epsilon \to 0$. This result is proved by considering the piecewise affine function $R_{(M,q)}(x) := \max\{x, Mx+q\}$ and without any assumption on the set of solvable qs. It will be interesting to see the connections between this result and results presented in our paper.

5. Appendix.

In this section we present the proofs of Lemma 4.5 and Theorem 4.4.

Let us recall (see [15]) that for a polyhedral multifunction F, Graph $F = \bigcup_{j=1}^{L} G_j$ where each G_j is a nonempty polyhedral set in $R^n \times R^m$. With Π_1 and Π_2 denoting the usual projections onto R^n and R^m, we have *dom* $F = \bigcup_{j=1}^{L} \Pi_1(G_j)$ and *ran* $F = \bigcup_{j=1}^{T} \Lambda_j$ where Λ_is are distinct and $\{\Lambda_1, \ldots, \Lambda_T\} = \{\Pi_2(G_j) : 1 \le j \le L\}$.

Before providing the proofs of Lemma 4.5 and Theorem 4.4, we record an equivalent formulation of the condition (*a*) of Theorem 4.4. The result formulated below may be considered as a global version (for a multifunction) of Proposition 2.1. It follows immediately from Prop. 1.4.4 of [4].

PROPOSITION 5.1. *Let F be a multifunction with dom $F \subseteq R^n$ and ran $F \subseteq R^m$. Suppose that Y is a nonempty subset of ran F and $X := F^{-1}(Y) := \{x : F(x) \cap Y \neq \emptyset\}$. Then the following are equivalent:*

(a) F^{-1} is lower semicontinuous on Y.

(a′) For any set E open in X, $F(E) \cap Y$ is open in Y.

Proof. (of Lemma 4.5.) First we describe β. For each j, the mapping $y \mapsto G_j \cap \Pi_2^{-1}(y)$ is Lipschitzian by a theorem of Walkup and Wets [45]. Since Π_1 is nonexpansive, it follows that $y \mapsto \Pi_1[G_j \cap \Pi_2^{-1}(y)]$ is also Lipschitzian; let β_j denote its Lipschitzian constant. We put $\beta := \max_j \beta_j$.

Assume that $p^* \neq q^*$, otherwise there is nothing to prove. Let $p^0 := p^*, x^0 \in F^{-1}(p^0)$, and Γ_0 be the collection of all G_js containing (x^0, p^0). Since Γ_0 is finite and all the G_js are closed, there exists an open set $U \times V$ in $R^n \times R^m$ such that

$$(x^0, p^0) \in U \times V \cap \{\, (x,p) : x \in F^{-1}(p),\, p \in Y \,\} \subseteq \bigcup_{G_j \in \Gamma_0} G_j \,.$$

This implies that

$$p^0 \in \Pi_2\left(\{\, (x,p) : x \in F^{-1}(p) \cap U,\, p \in V \cap Y\}\right) \subseteq \bigcup_{G_j \in \Gamma_0} \Pi_2(G_j)\,.$$

Thus,

$$p_0 \in (V \cap Y) \cap F(U \cap X) \subseteq \bigcup_{\Lambda_i \in \{ \Pi_2(G_j) \,:\, G_j \in \Gamma_0 \}} \Lambda_i \,.$$

Observe that condition (a) of Theorem 4.4 (assumed in the formulation of Lemma 4.5) implies, via Proposition 5.1, that the set $(V \cap Y) \cap F(U \cap X)$ is open in Y. Since $[p^0, q^*] \subseteq Y$, it follows that the open segment (p^0, q^*) must intersect at least one such Λ_i. Among all such Λ_i, we select one which contains the "maximal" subinterval of $[p^0, q^*]$. To be more specific, let t_0 denote the maximum of real numbers $t \in (0,1)$ with the property that the line segment $[p^0, (1-t)p^0 + tq^*]$ is contained in some Λ_i (which is of the form $\Pi_2(G_j)$ with $G_j \in \Gamma_0$). This t_0 is positive, and because Λ_is are closed and convex, there is some Λ_i which contains the interval $[p^0, (1-t_0)p^0 + t_0 q^*]$. Let Λ^0 denote such a Λ_i so that for some $G^0 \in \Gamma_0$ we have $\Lambda^0 = \Pi_2(G^0)$. The element $p^1 := (1-t_0)p^0 + t_0 q^* \in \Pi_2(G^0)$ and so the set $G^0 \cap \Pi_2^{-1}(p^1)$ is nonempty. Now obviously, $x^0 \in \Pi_1[G^0 \cap \Pi_2^{-1}(p^0)]$ and the inclusion

$$\Pi_1[G^0 \cap \Pi_2^{-1}(p^0)] \subseteq \Pi_1[G^0 \cap \Pi_2^{-1}(p^1)] + \beta \|p^1 - p^0\| B$$

shows that there exists an $x^1 \in \Pi_1[G^0 \cap \Pi_2^{-1}(p^1)]$ such that $\|x^0 - x^1\| \leq \beta \|p^0 - p^1\|$. Clearly, $(x^1, p^1) \in G^0$ and hence $p^1 \in F(x^1)$. (Note that $p^1 \in Y$, $x^1 \in X$ and of course, $(x^0, p^0) \in G^0$.) If $p^1 = q^*$, we are done. If $p^1 \neq q^*$, starting with x^1 and p^1, we repeat the above process to generate Γ_j, t_j, Λ^j, G^j, $p^{j+1} \in (p^0, q^*]$, and x^{j+1} for $j = 1, 2, \ldots$. We have $p^j \in Y$, $x^j \in F^{-1}(p^j)$, $\{(x^j, p^j), (x^{j+1}, p^{j+1})\} \subseteq G^j$ and $\|x^j - x^{j+1}\| \leq \beta \|p^j - p^{j+1}\|$ for each j. We now show that this process must stop in at most T steps. Since at each step the number t_j is positive, it follows from convexity that Λ^j is different from all the previous Λ^i $(0 \leq i \leq j-1)$. Thus at each step a new Λ is generated. Since there are at most T such Λs, the process must terminate in l steps with $l \leq T$. The inequalities

$$\|x^0 - x^l\| \leq \sum_{j=0}^{l-1} \|x^j - x^{j+1}\| \leq \beta \sum_{j=0}^{l-1} \|p^j - p^{j+1}\| = \beta \|p^* - q^*\|$$

prove

$$F^{-1}(p^*) \subseteq F^{-1}(q^*) + \beta \|p^* - q^*\| B \,.$$

This completes the proof of Lemma 4.5. □

Proof. (of Theorem 4.4.) Assume (a) and let $p, q \in Y$. There exist points $p_0 := p, \ldots, p_k := q$ such that $[p_i, p_{i+1}] \subseteq Y$ and $L(\gamma, p, q) \leq \alpha \|p - q\|$ where γ is a polygonal line joining p and q via $p_1, \ldots, p_{k-1}$. Let $x_0 \in F^{-1}(p_0)$; by Lemma 4.5 there exists $x_1 \in F^{-1}(p_1)$ such that

$$\|x_0 - x_1\| \leq \beta \|p_0 - p_1\| \,.$$

We repeat the same procedure for the intervals $[p_1, p_2], \cdots, [p_{k-1}, p_k]$ in order to get a sequence of points $x_j \in F^{-1}(p_j)$ such that $\|x_{j-1} - x_j\| \leq \beta \|p_{j-1} - p_j\|$ for $1 \leq j \leq k$. Hence,

$$\|x_0 - x_k\| \leq \sum_{j=1}^{k} \|x_{j-1} - x_j\| \leq \beta \sum_{j=1}^{k} \|p_{j-1} - p_j\| = \beta L(\gamma, p, q) \leq \alpha\beta \|p - q\| \,,$$

proving the inclusion $F^{-1}(p) \subseteq F^{-1}(q) + \alpha\beta||p-q||B$. Thus we have (b). □

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