Available online at www.sciencedirect.com

SCIENCE DIRECT

Linear Algebra and its Applications 416 (2006) 437–451

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

The **Q**-property of composite transformations and the **P**-property of Stein-type transformations on self-dual and symmetric cones

Roman Sznajder [a,*], M. Seetharama Gowda [b]

[a] *Department of Mathematics, Bowie State University, Bowie, MD 20715, United States*

[b] *Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, MD 21250, United States*

Received 12 April 2004; accepted 27 November 2005

Available online 10 January 2006

Submitted by H. Schneider

Abstract

Motivated by the **Q**-property of nonsingular M-matrices, Lyapunov and Stein transformations (corresponding to positive stable and Schur stable matrices, respectively) and their products [A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, 1994; M.S. Gowda, Y. Song, Math. Prog. 88 (2000) 575–587; M.S. Gowda, T. Parthasarathy, Linear Algebra Appl. 320 (2000) 131–144; M.S. Gowda, Y. Song, SIAM J. Matrix Anal. Appl. 24 (2002) 25–39], in the first part of the paper we present a unifying result on the product of **Q**-transformations defined on self-dual closed convex cones. The second part deals with the **P**-property of the linear transformation $I - S$ on a Euclidean Jordan algebra where S leaves the corresponding symmetric cone invariant and $\rho(S) < 1$. We prove the **P**-property for the Lorentz cone and present some partial results in the general case.

Keywords: Cone linear complementarity problem; Self-dual cone; Symmetric cone; Euclidean Jordan algebra; **Q**- and **P**-properties

* Corresponding author. Tel.: +1 301 860 3360; fax: +1 301 860 3364.

E-mail addresses: rsznajder@bowiestate.edu (R. Sznajder), gowda@math.umbc.edu (M.S. Gowda).

URL: http://www.math.umbc.edu/~gowda (M.S. Gowda).

0024-3795/$ - see front matter

doi:10.1016/j.laa.2005.11.021

1. Introduction

Consider a finite dimensional real Hilbert space H. Let K be a self-dual cone in H, that is,

$$K = K^* := \{y \in H : \langle y, x\rangle \geqslant 0 \text{ for all } x \in K\}.$$

Given a linear transformation $L : H \to H$ and a vector $q \in H$, the *cone linear complementarity problem*, LCP(L, K, q), is to find a vector $x^* \in H$ such that

$$x^* \in K, \quad L(x^*) + q \in K, \quad \text{and} \quad \langle L(x^*) + q, x^*\rangle = 0.$$

This is a particular case of variational inequality problem defined over a closed convex set and with respect to a nonlinear function on H. There is an extensive literature on linear (and nonlinear) complementarity problems and variational inequality problems. They appear in optimization, engineering, and various other fields (see [6]).

The present paper deals with the **Q**- and **P**- properties of certain types of linear transformations. A linear transformation L is said to have the **Q**-property on K if for all $q \in H$, LCP(L, K, q) has a solution; it is said to have the **P**-property on a Euclidean Jordan algebra if

$$x \text{ and } L(x) \text{ operator commute and } x \circ L(x) \leqslant 0 \;\Rightarrow\; x = 0$$

(see Section 2 for further details).

In the first part of the paper, we are concerned with the **Q**-property of a finite product of linear transformations. Our motivation comes from the following:

(1) Consider a nonsingular M-matrix (also known as a K-matrix), which is a matrix of the form $rI - A$ with $\rho(A) < r$. Such matrices have numerous properties (see, e.g., [2]). With respect to the usual inner product in R^n and the nonnegative cone R^n_+, a nonsingular M-matrix has the **Q**-property (equivalently, the **P**-property) [4]. It has been observed in [10] that any finite product of such matrices has the **Q**-property. The proof is based on the observation that the inverse of a nonsingular M-matrix is a (entry-wise) nonnegative matrix with positive diagonal.

(2) For a given matrix $A \in R^{n\times n}$, consider the Lyapunov transformation L_A defined on the space $\mathcal{S}^n$, of all real $n \times n$ symmetric matrices, by

$$L_A(X) = AX + XA^{\mathrm{T}}.$$

A celebrated result of Lyapunov (see Theorems 2.2.1 and 2.2.3 in [14]) states that

$$(L_A)^{-1}(\mathcal{S}^n_{++}) \subseteq \mathcal{S}^n_{++}$$

if and only if A is positive stable (which means that all eigenvalues of A lie in the open right-half plane). Here $\mathcal{S}^n_{++}$ is the interior of the closed convex cone $\mathcal{S}^n_+$ of positive semidefinite matrices in $\mathcal{S}^n$. It has been shown in [9] that these properties are equivalent to the **Q**-property (and also the **P**-property) of L_A. In [10], Gowda and Song prove, using degree-theoretic ideas, that any finite product of Lyapunov transformations corresponding to positive stable matrices has the **Q**-property with respect to $\mathcal{S}^n_+$.

(3) For a given matrix $A \in R^{n\times n}$, consider the Stein transformation S_A defined on the space $\mathcal{S}^n$ by

$$S_A(X) = X - AXA^{\mathrm{T}}.$$

The well-known result of Stein [19] says that

$$(S_A)^{-1}(\mathcal{S}^n_{++}) \subseteq \mathcal{S}^n_{++}$$

if and only if A is Schur stable (which means that all the eigenvalues of A lie in the open unit disk). In [8], Gowda and Parthasarathy show that these properties are further equivalent to the **Q**-property (and also the **P**-property) of S_A with respect to the semidefinite cone $\mathcal{S}^n_+$. In [10], Gowda and Song show that any finite product of Stein transformations corresponding to Schur stable matrices has the **Q**-property with respect to $\mathcal{S}^n_+$. The proof is based once again on degree-theoretic ideas.

(4) On $\mathcal{S}^n$, consider a set of transformations where each one is of the form $L = L_A - (I - S_B)$ with A and B real and square. Assuming the **P**-property for each one of these, Balaji [1] has recently shown that the product has the **Q**-property, extending the results of Gowda–Song and Gowda–Parthasarathy mentioned in items (2) and (3) above. Once again, the proof is degree-theoretic.

Seeing the commonality of the above results and that each of the above cones is a symmetric cone (see Section 2 for the definition), one may ask if a unifying result for the **Q**-property of the composite transformation can be established, say, on a Euclidean Jordan algebra. Our Theorem 6 (in Section 3) achieves this and much more: In addition to unifying the above results, our result is valid on any self-dual cone and its proof is independent of degree theoretic ideas. Moreover, it extends Corollary 10 in [8] dealing with a transformation of the form $I - S$ and prescribes a sufficient condition for the **Q**-property of a transformation of the form $L = R - S$ where R and S satisfy conditions of Schneider [17], namely, $S(K) \subseteq K$ and $R(K^\circ) \supseteq K^\circ$ or $R(K^\circ) \cap K^\circ = \emptyset$, where K° denotes the interior of K (see Lemma 7).

In the second part of the paper, we are concerned with the **P**-property of Stein-type transformation $L = I - S$ where we now assume that K is a symmetric cone in a Euclidean Jordan algebra with $S(K) \subseteq K$ and $\rho(S) < 1$. The **P**-property is obvious when K is the nonnegative orthant in R^n (as the matrix L is a nonsingular M-matrix). When K is $\mathcal{S}^n_+$ with $S = S_A$ (see item (3) above), the **P**-property has been verified in [8]. In Section 4, we prove the **P**-property of $I - S$ when K is the Lorentz cone, and present partial results in the general case.

2. Preliminaries

Throughout this paper, we assume that K is a (nonempty) closed convex cone in a finite dimensional real Hilbert space H. We further assume that K is self-dual, i.e.,

$$K = K^* := \{x \in H : \langle x, y\rangle \geqslant 0,\ \forall y \in K\}.$$

Then the interior of K, denoted by K°, is nonempty. A linear transformation $T : H \to H$ is said to be *copositive* (*strictly copositive*) on K if for all $0 \neq x \in K$, we have $\langle T(x), x\rangle \geqslant 0$ (respectively, > 0).

We recall the well-known result of Karamardian [16] specialized to T and K:

Theorem 1 (Karamardian's Theorem). *Suppose $T : H \to H$ is linear and for some $d \in K^\circ$, zero is the only solution of both* LCP$(T, K, 0)$ *and* LCP(T, K, d). *Then T has the* **Q**-*property with respect to K.*

It easily follows from this result that *if a linear transformation is strictly copositive on K, then it has the* **Q**-*property on K.*

For a linear transformation L on H, $\rho(L)$ denotes the spectral radius of L. We recall that

$$\rho(L) = \lim_{n\to\infty} \|L^n\|^{1/n}.$$

A nonempty convex subset F of K is a *face* of K if

$$0 < t < 1,\ \ u, v \in K,\ \ tu + (1-t)v \in F \ \ \Rightarrow\ \ u, v \in F.$$

In this case, we write

$$F \lhd K.$$

It is easily verified that every face of K is a closed convex cone. Given $x \in K$, the intersection of all faces of K containing x is itself a face—called the *face generated by* x. For any closed convex cone G contained in K, we define the *complementary cone* corresponding to G by

$$G^{\Delta} := G^{\perp} \cap K = \{y \in K : y \perp G\},$$

where we write $u \perp v$ to mean $\langle u, v\rangle = 0$.

We need the following:

Lemma 2. *Suppose $x, y \in K$ such that $x \perp y$. Let F be the face generated by y. Then $x \in F^{\Delta}$.*

This lemma is well-known. It easily follows from the observation that the set $E := \{z \in F : z \perp x\}$ is a face of K containing y.

The cone K is said to be *facially exposed* (see Theorem 6.7 in [3]) if

$$(F^{\Delta})^{\Delta} = F$$

for all $F \lhd K$.

2.1. Euclidean Jordan algebras

In this subsection, we briefly recall some concepts, properties, and results from Euclidean Jordan algebras. Most of these can be found in [5,18,12].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle\cdot,\cdot\rangle)$ where $(V, \langle\cdot,\cdot\rangle)$ is a finite dimensional inner product space over R and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in V$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$ where $x^2 := x \circ x$, and
(iii) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all $x, y, z \in V$.

In addition, we assume that there is an element $e \in V$ (called the *unit* element) such that $x \circ e = x$ for all $x \in V$.

In a Euclidean Jordan algebra V, the set of squares

$$K := \{x \circ x : x \in V\}$$

is a *symmetric cone* (see [5, p. 46]). This means that K is a self-dual cone and for any two elements $x, y \in \text{int}(K)$, there exists an invertible linear transformation $\Gamma : V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$. If $x \in K$, we write $x \geqslant 0$ and write $y \leqslant 0$ to mean $-y \geqslant 0$.

For $x \in V$, we define

$$m(x) := \min\{k > 0 : \{e, x, \ldots, x^k\} \text{ is linearly dependent}\}$$

and *rank* of V by $r = \max\{m(x) : x \in V\}$. An element $c \in V$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, e_2, \ldots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \ \text{ if } i \neq j \quad \text{and} \quad \sum_1^m e_i = e.$$

Note that $\langle e_i, e_j\rangle = \langle e_i \circ e_j, e\rangle = 0$ whenever $i \neq j$.

Theorem 3 (The spectral decomposition theorem [5]). *Let V be a Euclidean Jordan algebra with rank r. Then for every $x \in V$, there exists a Jordan frame $\{e_1, \dots, e_r\}$ and real numbers $\lambda_1, \dots, \lambda_r$ such that*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{1}$$

The expression $\lambda_1 e_1 + \cdots + \lambda_r e_r$ is the *spectral decomposition* (*or the spectral expansion*) of x.

We note that

$$x \geqslant 0 \text{ (i.e., } x \in K) \;\Rightarrow\; 0 \leqslant \langle x, e_i \rangle = \lambda_i \|e_i\|^2 \;\Rightarrow\; \lambda_i \geqslant 0.$$

The Peirce decomposition

Fix a Jordan frame $\{e_1, e_2, \dots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, 2, \dots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = R e_i$$

and when $i \neq j$,

$$V_{ij} := \left\{x \in V : x \circ e_i = \tfrac{1}{2} x = x \circ e_j\right\}.$$

Then we have the following.

Theorem 4 [5, Theorem IV.2.1]. *The space V is the orthogonal direct sum of spaces V_{ij} $(i \leqslant j)$. Furthermore*,

$$\begin{aligned} &V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj}, \\ &V_{ij} \circ V_{jk} \subset V_{ik} \quad \text{if } i \neq k, \\ &V_{ij} \circ V_{kl} = \{0\} \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset. \end{aligned}$$

Thus, given any Jordan frame $\{e_1, e_2, \dots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij},$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$.

The algebra $\mathcal{S}^n$

Let $\mathcal{S}^n$ be the set of all $n \times n$ real symmetric matrices with the inner and Jordan product given by

$$\langle X, Y \rangle := \operatorname{trace}(XY) \quad \text{and} \quad X \circ Y := \tfrac{1}{2}(XY + YX).$$

In this setting, the cone of squares $\mathcal{S}^n_+$ is the set of all positive semidefinite matrices in $\mathcal{S}^n$.

The Cartesian product of S^1s will give us the *Euclidean Jordan algebra R^n* where the inner product is the usual inner product and the Jordan product is the componentwise product of vectors.

The algebra $\mathscr{L}^n$

Consider R^n $(n > 1)$ where any element x is written as

$$x = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}$$

with $x_0 \in R$ and $\bar{x} \in R^{n-1}$. The inner product in R^n is the usual inner product. The Jordan product $x \circ y$ in R^n is defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \bar{y} \end{bmatrix} := \begin{bmatrix} \langle x, y \rangle \\ x_0\bar{y} + y_0\bar{x} \end{bmatrix}.$$

We shall denote this Euclidean Jordan algebra $(R^n, \circ, \langle \cdot, \cdot \rangle)$ by $\mathscr{L}^n$. In this algebra, the cone of squares, denoted by $\mathscr{L}^n_+$, is called the *Lorentz cone* (or the second-order cone). It is given by

$$\mathscr{L}^n_+ = \{x : \|\bar{x}\| \leqslant x_0\}.$$

The unit element in $\mathscr{L}^n$ is $e = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. We note the spectral decomposition of any x with $\bar{x} \neq 0$:

$$x = \lambda_1 e_1 + \lambda_2 e_2,$$

where

$$\lambda_1 := x_0 + \|\bar{x}\|, \quad \lambda_2 := x_0 - \|\bar{x}\|$$

and

$$e_1 := \frac{1}{2}\begin{bmatrix} 1 \\ \frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix} \quad \text{and} \quad e_2 := \frac{1}{2}\begin{bmatrix} 1 \\ -\frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}.$$

In a Euclidean Jordan algebra V, for a given $x \in V$, we define the corresponding *Lyapunov transformation* $L_x : V \to V$ by

$$L_x(z) = x \circ z.$$

We say that elements x and y *operator commute* if L_x and L_y commute, i.e.,

$$L_x L_y = L_y L_x.$$

It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame ([5, Lemma X.2.2] or [18, Theorem 27]). In the case of $\mathscr{S}^n$, matrices X and Y operator commute if and only if $XY = YX$. In the case of $\mathscr{L}^n$, vectors x and y (see Example 2) operator commute if and only if either $\bar{y}$ is a multiple of $\bar{x}$ or $\bar{x}$ is a multiple of $\bar{y}$.

Simple Jordan algebras and the structure theorem

A Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two Euclidean Jordan algebras. The classification theorem [5, Chapter V] says that every simple Euclidean Jordan algebra is isomorphic to one of the following:

(1) The algebra $\mathscr{S}^n$ of $n \times n$ real symmetric matrices.
(2) The algebra $\mathscr{L}^n$.
(3) The algebra $\mathscr{H}_n$ of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(4) The algebra $\mathscr{Q}_n$ of all $n \times n$ quaternion Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

(5) The algebra $\mathcal{O}_3$ of all 3×3 octonion Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

The following result characterizes all Euclidean Jordan algebras.

Theorem 5 [5, Propositions III.4.4 and III.4.5, Theorem V.3.7]. *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

Faces of a symmetric cone can be described easily: If $F \lhd K$ and K is the symmetric cone in V, then there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and $1 \leqslant k \leqslant r$ such that

$$F = \{x \in K : x \circ (e_1 + e_2 + \cdots + e_k) = x\}$$

(see [11]). Given F as above, it can be easily shown (e.g., using the Peirce decomposition and Proposition 3.2 in [11]) that

$$F^{\Delta} = \{z \in K : z \circ (e_{k+1} + \cdots + e_r) = z\}.$$

From this it follows that

$$(F^{\Delta})^{\Delta} = F \quad \text{for all } F \lhd K. \tag{2}$$

Thus, every symmetric cone is facially exposed. See Theorem 6 in [20] for a result of this type for homogeneous cones (which include symmetric cones).

3. The Q-property

Recall that K is a self-dual cone in a finite dimensional real Hilbert space H and $L : H \to H$ has the **Q**-property if for all $q \in H$, LCP(L, K, q) has a solution. We begin by noting that the **Q**-property is not product invariant: In $H = R^n$ with $K = R^n_+$, take

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

A and B are **P**-matrices (that is, all principal minors are positive) and hence are **Q**-matrices [4]. However, it is easily seen that the product AB does not have the **Q**-property. Thus, in general, the product of transformations having the **Q**-property need not have the **Q**-property. In Corollary 9 below, we specify sufficient conditions for the product to have the **Q**-property. First, we present our main theorem of this section. This theorem and its corollaries are motivated by several results (see items (1)–(4)) mentioned in Section 1.

Theorem 6. *Suppose $L : H \to H$ is an invertible linear transformation satisfying the following*:

(a) $L^{-1}(K) \subseteq K$ *and*
(b) $L(F) \cap K \subseteq F$ *for all* $F \lhd K$.

Then L^{-1} is strictly copositive on K. Hence L^{-1} and L have the **Q**-*property.*

Proof. Since K is self-dual, by (a), L^{-1} is copositive on K. Suppose, if possible, $\langle L^{-1}x, x\rangle = 0$ for some $0 \neq x \in K$. Letting $y = L^{-1}x$, we see that $0 \neq y \in K$ (by (a)) and $\langle x, y\rangle = 0$. Let F be the face generated by y. Then $x = L(y) \in L(F) \cap K$ and so by (b), $x \in F$. On the other hand, by Lemma 2, $x \in F^{\Delta}$. Hence $x = 0$, yielding a contradiction. Hence L^{-1} is strictly copositive on K. By Theorem 1, L^{-1} has the **Q**-property on K. Now for any $q \in H$, let x be a solution of LCP($L^{-1}, K, -L^{-1}q$), so that $x \in K$, $y = L^{-1}(x) - L^{-1}q \in K$, with $\langle x, y\rangle = 0$. Then $y \in K$, $x = L(y) + q \in K$, and $\langle y, x\rangle = 0$. It follows that y solves LCP(L, K, q), proving the **Q**-property of L on K. □

Remarks

(i) In the above result, L is assumed to be invertible. Because of this, condition (a) is equivalent to: $L^{-1}(K^\circ) \subseteq K^\circ$.

(ii) Theorem 6 deals with an invertible transformation. It is possible to state a similar result for general transformations in the following way: Let M be linear on H with $M(K) \subseteq K$, and for all $F \lhd K$,

$$x \in F, x = My, y \in K \Rightarrow y \in F.$$

Then M is strictly copositive on K and hence has the **Q**-property on K. This can be seen as follows: From $M(K) \subseteq K$, we get the copositivity of M on K. If M is not strictly copositive on K, then there is a nonzero $x \in K$ such that $\langle Mx, x\rangle = 0$. Let $z = Mx \in K$ (because $M(K) \subseteq K$). Let F be the face of K generated by z. We have $z, x \in K$, $\langle z, x\rangle = 0$; hence $x \in F^{\perp}$. Then $z \in F$, $z = Mx$, $x \in K \Rightarrow x \in F$. Thus $x = 0$, which is a contradiction. So M is strictly copositive and by Theorem 1 we get the **Q**-property of M on K.

By unifying and extending the results of Lyapunov and Stein (mentioned in Section 1), Schneider [17] proves the following.

Lemma 7 (Schneider [17]). *Consider linear transformations R and S satisfying conditions $S(K) \subseteq K$ and $R(K^\circ) \supseteq K^\circ$ or $R(K^\circ) \cap K^\circ = \emptyset$. Let $L = R - S$. Then the following are equivalent*:

(i) *R is invertible, $R^{-1}(K) \subseteq K$, and $\rho(R^{-1}S) < 1$.*
(ii) *L is invertible and $L^{-1}(K^\circ) \subseteq K^\circ$.*
(iii) *There exists a vector $d \in K^\circ$ such that $L(d) \in K^\circ$.*

In [8], Gowda and Parthasarathy show that when R is a multiple of the Identity, the above conditions are further equivalent to the **Q**-property of L. They also observe (Cf. [8, Example 2]) that the **Q**-property does not hold for a general L. In the result below, we prescribe a sufficient condition on R that yields the **Q**-property of L.

Corollary 8. *Suppose R and S are linear on H, $L = R - S$, $S(K) \subseteq K$,*

(α) *L is invertible, $L^{-1}(K) \subseteq K$, and*
(β) *$R(F) \cap K \subseteq F$ for all faces F in K.*

Then conditions (a) *and* (b) *of Theorem 6 hold. Hence L has the* **Q***-property.*

Proof. We need only show that $L(F) \cap K \subseteq F$ for all faces F in K. To see this, suppose F is a face of K and let $y \in L(F) \cap K$. Let $y = L(x) \in K$ for some $x \in F$. Then $y = R(x) - S(x)$

implies that $R(x) = y + S(x) \in K$ and so $R(x) \in R(F) \cap K$. By our assumption (β), $R(x) \in F$. Since F is a face and $R(x) = y + S(x)$ with $y, S(x) \in K$, we have $y \in F$, proving condition (b) of the theorem. □

Remark. When L has the **Q**-property on K, item (iii) in Schneider's Lemma is satisfied. (This can be seen by noting that if x is a solution of LCP$(L, K, -q)$ where $q \in K^\circ$, then $L(x) - q \in K$. Hence $L(x) \in q + K^\circ \subseteq K^\circ$. Perturbing x, we get a vector d with $d \in K^\circ$ such that $L(d) \in K^\circ$.) Thus, when $R(F) \cap K \subseteq F$ for all faces F in K, items (i)–(iii) are further equivalent to the **Q**-property of L.

Corollary 9. *Suppose $L_i : H \to H$ $(i = 1, 2, \ldots, n)$ satisfy conditions* (a) *and* (b) *of Theorem 6. Then the product $\prod L_i$ will also satisfy the same conditions. Hence the product has the* **Q***-property.*

Proof. We will prove the result by induction on n. By Theorem 6, the result is true for $n = 1$. We assume the result for $n - 1$. Assume that $L_i : H \to H$ $(i = 1, 2, \ldots, n)$ satisfy conditions (a) and (b) of the above theorem. Clearly the product $\prod_1^n L_i$ satisfies condition (a), so we have only to prove condition (b). To this end, let $x \in F \triangleleft K$ and $z = \prod_1^n L_i(x) \in K$. We have to show that $z \in F$. Now, $z = \left(\prod_1^{n-1} L_i\right)(L_n(x)) \in K$. Hence $L_n(x) = \left(\prod_1^{n-1} L_i\right)^{-1}(z) \in K$. Therefore, $L_n(x) \in L_n(F) \cap K \subset F$. Hence $z = \left(\prod_1^{n-1} L_i\right)(L_n(x)) \in \left(\prod_1^{n-1} L_i\right)(F) \cap K \subseteq F$ by the induction hypothesis. So we have proved that $z \in F$. Thus by induction, the result holds. □

Example 1. Consider $H = \mathscr{S}^n$, $R = L_A$, $S = 0$, where A is positive stable. Then $L_A^{-1}(\mathscr{S}^n_+) \subseteq \mathscr{S}^n_+$ by Lyapunov's Theorem. We verify condition (β) of Corollary 8. Given a face F of $\mathscr{S}^n_+$, take $X \in F$ such that $L_A(X) \in \mathscr{S}^n_+$. We will show that $L_A(X) \in F$. By Theorem 3.6 in [13], there is an index $1 \leqslant m \leqslant n$ and an orthogonal matrix $U \in R^{n\times n}$ such that

$$X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{T}},$$

where $X_1 \in \mathscr{S}^m_+$. Certainly,

$$U^{\mathrm{T}} L_A(X) U = \begin{bmatrix} L_{B_1}(X_1) & C \\ C^{\mathrm{T}} & 0 \end{bmatrix},$$

where $B = U^{\mathrm{T}} A U$ and B_1 is the block of B compatible with X_1. Since $U^{\mathrm{T}} L_A(X) U \in \mathscr{S}^n_+$, we must have $C = 0$. This implies (by Theorem 3.6 in [13]), that

$$L_A(X) = U \begin{bmatrix} L_{B_1}(X_1) & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{T}} \in F.$$

Now the conditions of Corollary 8 are satisfied. Hence L_A has the **Q**-property. It follows from Corollary 9 that any finite product of such transformations will also have the **Q**-property. Thus we have recovered Theorem 9 in [10].

Example 2. On any real Hilbert space H, let $R = rI$, $S(K) \subseteq K$, and $\rho(S) < r$. Then $L = rI - S$ is invertible and $(rI - S)^{-1} = \sum_0^\infty \frac{S^k}{r^{k+1}}$. Because $S(K) \subseteq K$, we have $L^{-1}(K) \subseteq K$. Thus condition (α) of Corollary 8 is satisfied. Condition (β) is obvious. Hence conditions (a) and (b) of Theorem 6 hold. By previous Corollary, any finite product of transformations of the form $rI - S$ will have the **Q**-property. This extends Corollary 10 in [8] (which is stated for a single transformation of the form $rI - S$) and Theorem 10 in [10] (which is stated for Stein transformations of the form $S_A(X) = X - AXA^{\mathrm{T}}$ on $\mathscr{S}^n$).

Corollary 10. *Suppose that K is facially exposed. Let R and S be linear on H, $L = R - S$, with $S(K) \subseteq K$ and R satisfying the following*:

- *For any $F \lhd K$, $R(F) \subseteq (F^{\Delta})^{\perp}$.*
- *L is invertible, and*
- *For all $q \in K^{\circ}$, $\mathrm{LCP}(L, K, -q)$ has a solution.*

Then L satisfies conditions of Corollary 8 and so L has the **Q***-property. Moreover, any finite product of such transformations will have the* **Q***-property.*

Proof. We first verify condition (α). Fix $q \in K^{\circ}$ and let x be a solution of $\mathrm{LCP}(L, K, -q)$. Then x and $y := L(x) - q$ are both in K and are orthogonal to each other. Let F be the face of K generated by x in which case, $y \in F^{\Delta}$ (by Lemma 2). By the imposed condition on R, we have $R(x) \in (F^{\Delta})^{\perp}$ and so $\langle R(x), y\rangle = 0$. As q and $S(x)$ are both in $K = K^*$, we get

$$0 \leqslant \langle y, y\rangle = \langle R(x), y\rangle - \langle S(x) + q, y\rangle \leqslant 0$$

and $y = 0$. Hence $L(x) = q$ and $L^{-1}(q) = x \in K$. Thus $L^{-1}(K^{\circ}) \subseteq K$, which implies, by continuity of L, the inclusion $L^{-1}(K) \subseteq K$. To see (β), let $F \lhd K$. Then $R(F) \cap K \subseteq (F^{\Delta})^{\perp} \cap K = (F^{\Delta})^{\Delta}$. Since K is assumed to facially exposed, we have $(F^{\Delta})^{\Delta} = F$. Thus we have $R(F) \cap K \subseteq F$, proving (β). We conclude that L has the **Q**-property and moreover, any finite product of such transformations will have the **Q**-property. □

Example 3. Consider $H = \mathcal{S}^n$, $R = L_A$, $S = I - S_B$. As in Example 1, it can be easily shown that for any $F \lhd K$, $L_A(F) \subseteq (F^{\Delta})^{\perp}$. If $L = L_A - (I - S_B)$ has the **P**-property, then the two remaining conditions of the previous corollary hold. Thus L has the **Q**-property and any finite product of such transformations will also have the **Q**-property. Thus we recover a result of Balaji [1] mentioned in Section 1.

We conclude this section by presenting two variations of Corollary 9. In the proof of Corollary 9, $L^{-1} = L_n^{-1} C$ where $C = \prod_{n-1}^{1} L_k^{-1}$ is strictly copositive on K. By assuming $L_n = rI - S$, we present the following results.

Proposition 11. *Suppose $T = (rI - S)^{-1} C$ where $S(K) \subseteq K$, $\rho(S) < r$, C is strictly copositive on K with $C(K) \subseteq K$. Then T is strictly copositive on K.*

Proof. Without loss of generality, let $r = 1$. As K is invariant under both S and C, from $T = \left(\sum_0^{\infty} S^k\right) C$, we get $T(K) \subseteq K$. Now let $0 \neq x \in K$ in which case, $y = Tx = (I - S)^{-1} Cx \in K$. Then $(I - S)y = Cx$ and so $y = Sy + Cx$. Therefore $\langle Tx, x\rangle = \langle y, x\rangle = \langle Sy, x\rangle + \langle Cx, x\rangle > 0$ as $Sy \in K$ and C is strictly copositive. □

By dropping the condition $C(K) \subseteq K$ in the above result, we get

Proposition 12. *Suppose $T = (rI - S)^{-1} C$ where $S(K) \subseteq K$, $\rho(S) < r$ and C is strictly copositive on K. Then T has the* **Q***-property.*

Proof. Without loss of generality, let $r = 1$. By Schneider's Lemma, there is a $d \in K^{\circ}$ such that $(I - S)d \in K^{\circ}$. To prove the **Q**-property of T, we show that zero is the only solution of

LCP(T, td) for any $t \geqslant 0$ and then appeal to Theorem 1. To this end, fix any $t \geqslant 0$ in R, and let x be a nonzero solution of LCP(T, td). Put $y = Tx + td$ so that $x, y \in K$ and $\langle x, y\rangle = 0$. Then

$$\langle (I - S)y, x\rangle = \langle (I - S)Tx, x\rangle + t\langle (I - S)d, x\rangle \geqslant \langle Cx, x\rangle > 0,$$

as $(I - S)d \in K^{\circ}$ and C is strictly copositive on K. On the other hand,

$$\langle (I - S)y, x\rangle = \langle y, x\rangle - \langle Sy, x\rangle \leqslant 0$$

as $\langle y, x\rangle = 0$ and $S(K) \subseteq K$. We reach a contradiction. Hence zero is the only solution of LCP(T, td). This completes the proof. □

Remark. A result of the above type has been recently observed by Isac and Németh [15, Theorem 6.6] for the standard linear complementarity problem, i.e., when $H = R^n$ and $K = R^n_+$, based on an existence result for nonlinear complementarity problems.

4. The P-property

Recall that a linear transformation L on a Euclidean Jordan algebra V is said to have the **P**-property if

$$x \text{ and } L(x) \text{ operator commute and } x \circ L(x) \leqslant 0 \quad\Rightarrow\quad x = 0,$$

where $z \leqslant 0$ means that $-z \in K$. The above **P**-property is a generalization of the familiar **P**-property of a matrix [7]. As shown in [12], the **P**-property is implied by numerous other properties such as: the globally uniquely solvable property (which means that for all $q \in V$, LCP(L, K, q) has a unique solution), the Jordan **P**-property, the order **P**-property, strong monotonicity property, etc. Moreover, the **P**-property implies the **Q**-property.

It is known, see [9,8], that when $V = \mathcal{S}^n$, L_A has the **P**-property if and only if A is positive stable and S_A has the **P**-property if and only if A is Schur stable. Motivated by the latter result, we may ask if, in general,

$$I - S \text{ has the } \mathbf{P}\text{-property when } S(K) \subseteq K \text{ and } \rho(S) < 1.$$

In this section, we answer this completely when $V = \mathcal{L}^n$ and present some partial results in the general case.

Proposition 13. *Suppose $V = \mathcal{L}^n$ and $S(\mathcal{L}^n_+) \subseteq \mathcal{L}^n_+$ and $\rho(S) < 1$. Then $I - S$ has the* **P**-*property.*

Proof. Suppose $x \neq 0$ and $(I - S)x$ operator commute with $x \circ (I - S)x \leqslant 0$. Then we may write

$$0 \neq x = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad S(x) = (\lambda_1 - \mu_1)e_1 + (\lambda_2 - \mu_2)e_2,$$

where $\{e_1, e_2\}$ is a Jordan frame in $\mathcal{L}^n$ and $\lambda_i\mu_i \leqslant 0$, $i = 1, 2$.

We consider several cases.

Case 1. $\lambda_1 \neq 0$, $\lambda_2 = 0$. (Note: The case $\lambda_1 = 0$, $\lambda_2 \neq 0$ is similar.)

Let

$$\gamma_1 := 1 - \frac{\mu_1}{\lambda_1}.$$

Then $x = \lambda_1 e_1$ and $S(x) = \lambda_1\gamma_1 e_1 - \mu_2 e_2$. We see that

$$S(e_1) = \gamma_1 e_1 - \frac{\mu_2}{\lambda_1} e_2. \tag{3}$$

Since $S(\mathscr{L}^n_+) \subseteq \mathscr{L}^n_+$, $S(e_1) \geqslant 0$ and hence $-\frac{\mu_2}{\lambda_1} \geqslant 0$. We now claim that

$$\langle S^n(e_1), e_1\rangle \geqslant \|e_1\|^2\gamma_1^n, \quad \forall n = 1, 2, \ldots. \tag{4}$$

Eq. (3) shows that the above statement is true for $n = 1$. Assume that the above statement is true for k so that $\langle S^k(e_1), e_1\rangle \geqslant \|e_1\|^2\gamma_1^k$. From (3),

$$S^{k+1}(e_1) = \gamma_1 S^k(e_1) - \frac{\mu_2}{\lambda_1} S^k(e_2).$$

As $S(\mathscr{L}^n_+) \subseteq \mathscr{L}^n_+$, $S^k e_2 \geqslant 0$. Hence we have, from $-\frac{\mu_2}{\lambda_1} \geqslant 0$,

$$\begin{aligned}\langle S^{k+1}(e_1), e_1\rangle &= \gamma_1\langle S^k(e_1), e_1\rangle - \frac{\mu_2}{\lambda_1}\langle S^k(e_2), e_2\rangle\\ &\geqslant \gamma_1\langle S^k(e_1), e_1\rangle\\ &\geqslant \gamma_1^{k+1}\|e_1\|^2.\end{aligned}$$

Hence by induction, we have the claim. As $\rho(S) < 1$, $S^n \to 0$. Letting $n \to \infty$ in (4), we get $\gamma_1 < 1$. But $\gamma_1 = 1 - \frac{\mu_1}{\lambda_1} \geqslant 1$ is a contradiction. Hence, this case cannot happen.

Case 2. $\lambda_1 \neq 0$ $\lambda_2 \neq 0$.

Put $\gamma_1 := 1 - \frac{\mu_1}{\lambda_1}$ and $\gamma_2 := 1 - \frac{\mu_2}{\lambda_2}$ and note that $\gamma_1, \gamma_2 \geqslant 1$. Now

$$x = \lambda_1 e_1 + \lambda_2 e_2 \quad\text{and}\quad S(x) = \lambda_1\gamma_1 e_1 + \lambda_2\gamma_2 e_2.$$

Then $S^k(x) = \lambda_1 S^k(e_1) + \lambda_2 S^k(e_2)$. Hence

$$\begin{aligned}S^{k+1}(x) &= \lambda_1\gamma_1 S^k(e_1) + \lambda_2\gamma_2 S^k(e_2)\\ &= \gamma_1[S^k(x) - \lambda_2 S^k(e_2)] + \lambda_2\gamma_2 S^k(e_2)\\ &= \gamma_1 S^k(x) + \lambda_2(\gamma_2 - \gamma_1) S^k(e_2)\end{aligned} \tag{5}$$

and by symmetry,

$$S^{k+1}(x) = \gamma_2 S^k(x) + \lambda_1(\gamma_1 - \gamma_2) S^k(e_1). \tag{6}$$

Subcase 2.1. $\lambda_1 > 0$ and $\lambda_2 > 0$. (Note: The case $\lambda_1 < 0$ and $\lambda_2 < 0$ can be handled by working with $-x$.)

First suppose that $\gamma_2 - \gamma_1 \geqslant 0$. Then by (5), $\langle S^{k+1}(x), e_1\rangle \geqslant \gamma_1\langle S^k(x), e_1\rangle$. This implies that

$$\langle S^{n+1}(x), e_1\rangle \geqslant \gamma_1^{n+1}\lambda_1\|e_1\|^2$$

for all natural numbers. Now letting $n \to \infty$ and using the fact that $S^n \to 0$, we see that $\gamma_1 < 1$. This is a contradiction since $\gamma_1 \geqslant 1$.

If $\gamma_2 - \gamma_1 < 0$, then $\gamma_1 - \gamma_2 > 0$. In this case, we use (6) and proceed as before to get a contradiction.

Subcase 2.2. $\lambda_1 > 0$ and $\lambda_2 < 0$. (Note: The case $\lambda_1 < 0$ and $\lambda_2 > 0$ can be handled by working with $-x$.)

First suppose that $\gamma_2 - \gamma_1 \geqslant 0$. From (5) we have

$$S^{k+1}(-x) = \gamma_1 S^k(-x) - \lambda_2[\gamma_2 - \gamma_1] S^k(e_2).$$

Since $\lambda_2 < 0$ and $S^k(e_2) \geqslant 0$, we have $\langle S^{k+1}(-x), e_2\rangle \geqslant \gamma_1 \langle S^k(-x), e_2\rangle$. This yields

$$\langle S^{n+1}(-x), e_2\rangle \geqslant \gamma_1^n(-\lambda_2)\gamma_2\|e_2\|^2$$

for all natural numbers. Letting $n \to \infty$ and using the fact that $S^n \to 0$, we see that $\gamma_1 < 1$. This clearly is a contradiction.

If $\gamma_2 - \gamma_1 < 0$, we use (6) to get

$$\langle S^{n+1}(x), e_1\rangle \geqslant \gamma_2^n \lambda_1 \gamma_1 \|e_1\|^2.$$

We proceed as before to get a contradiction.

Thus in all cases, we arrive at a contradiction. We see that $x = 0$, proving the **P**-property of $I - S$. □

Remark. It should be noted that any Euclidean Jordan algebra of rank 2 is isomorphic to some $\mathscr{L}^n$; in particular, $\mathscr{S}^2$ is isomorphic to $\mathscr{L}^3$ (see [5, p. 66]).

We now turn our attention to a general Euclidean Jordan algebra and present some partial results.

Proposition 14. *Suppose $S : V :\to V$ be linear with $\|S\| \leqslant 1$ and $\rho(S) < 1$. Then $I - S$ has the* **P**-*property.*

Proof. Suppose that $0 \neq x$ and $(I - S)x$ operator commute and $x \circ (x - S(x)) \leqslant 0$. We write $x = \sum \lambda_i e_i$ and $S(x) = \sum(\lambda_i - \mu_i)e_i$ where $\lambda_i\mu_i \leqslant 0$ for all i. Then

$$\|x\|^2 = \langle x \circ x, e\rangle \leqslant \langle x \circ S(x), e\rangle = \langle x, S(x)\rangle \leqslant \|x\|\|S(x)\| \leqslant \|x\|^2.$$

So equality holds in the Cauchy–Schwarz inequality, and thus, $S(x) = \alpha x$ for some number $\alpha \geqslant 0$. Then $\lambda_i - \mu_i = \alpha\lambda_i$ for all i. Since $x \neq 0$, there is some λ_i which is nonzero. For this λ_i, $1 - \frac{\mu_i}{\lambda_i} = \alpha$. Now $\lambda_i\mu_i \leqslant 0 \Rightarrow \frac{\mu_i}{\lambda_i} \leqslant 0 \Rightarrow \alpha \geqslant 1$. Note that α is an eigenvalue of S. Thus $\rho(S) \geqslant 1$ contradicting the assumption. Hence $x = 0$, proving the **P**-property of $I - S$. □

Corollary 15. *Suppose $S : V \to V$ be linear, self-adjoint, and $\rho(S) < 1$. Then $I - S$ has the* **P**-*property.*

Proof. For a self-adjoint transformation, the spectral radius and (operator) norm coincide. The corollary now follows from the previous proposition. □

We next consider rank one transformations.

Proposition 16. *Let $u, v \in K$, and define $S : V \to V$ by $S(x) := u\langle v, x\rangle$. If $\rho(S) < 1$, then $I - S$ has the* **P**-*property.*

Proof. From $S(x) := u\langle v, x\rangle$, we get $S^n(x) = u\langle u, v\rangle^{n-1}\langle v, x\rangle$. This gives

$$\|S^n\| = \|u\||\langle u, v\rangle|^{n-1}\|v\|.$$

Then

$$\rho(S) = \lim \|S^n\|^{1/n} = |\langle u, v\rangle|.$$

Now assume that x commutes with $x - S(x)$ and $x \circ [x - S(x)] \leqslant 0$. Then there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ such that

$$x = \sum x_i e_i \quad \text{and} \quad y = x - S(x) = \sum y_i e_i.$$

In addition, $x_i y_i \leqslant 0$ for all i. With respect to the Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we have the Peirce decomposition

$$v = \sum v_i e_i + \sum_{i<j} v_{ij}.$$

Letting

$$\theta_i := \|e_i\|^2,$$

we have by the orthogonality properties of V_{ij}'s, $S(x) = u\langle v, x\rangle = u \sum v_i x_i \theta_i$. Hence,

$$\sum y_i e_i = y = x - S(x) = \sum x_i e_i - u \sum v_i x_i \theta_i.$$

Case 1. $\sum v_i x_i \theta_i = 0$. Then $x_i = y_i$ and $x_i y_i \leqslant 0$ for all i. Hence, $x_i^2 \leqslant 0$, so $x_i = 0$ for all i and, consequently, $x = 0$.

Case 2. $\sum v_i x_i \theta_i \neq 0$. Then

$$u = \frac{\sum x_i e_i - \sum y_i e_i}{\sum x_i v_i \theta_i},$$

so $|\langle u, v\rangle| < 1$ implies

$$\left|1 - \frac{\sum v_i y_i \theta_i}{\sum x_i v_i \theta_i}\right| < 1.$$

Subcase 2.1. $\sum v_i x_i \theta_i > 0$. Then $u \in K$ implies that $\sum x_i e_i - \sum y_i e_i \in K$, so $x_i \geqslant y_i$ for all i. Because $x_i y_i \leqslant 0$, $y_i \leqslant 0 \leqslant x_i$ for all i. As $v \in K$, we have $\sum v_i y_i \theta_i \leqslant 0 \leqslant \sum v_i x_i \theta_i$, and so

$$\left|1 - \frac{\sum v_i y_i \theta_i}{\sum x_i v_i \theta_i}\right| \geqslant 1.$$

This is clearly a contradiction.

Subcase 2.2. $\sum v_i x_i \theta_i < 0$. Reversing the roles of x_i and y_i in Subcase 2.1 we get $x_i \leqslant 0 \leqslant y_i$ for all i. Thus, $\sum v_i x_i \theta_i \leqslant 0 \leqslant \sum v_i y_i \theta_i$. Again,

$$\left|1 - \frac{\sum v_i y_i \theta_i}{\sum x_i v_i \theta_i}\right| \geqslant 1,$$

yielding a contradiction.

Thus, Case 2 cannot happen. By Case 1, $x = 0$, so the transformation $I - S$ has the **P**-property. □

5. Concluding remarks

In this paper, we have presented a (unifying) result on the **Q**-property of a finite product of linear transformations on a self-dual cone. Also, the **P**-property of the transformation $I - S$ on a symmetric cone is studied. When S is linear with $S(K) \subseteq K$ and $\rho(S) < 1$, the **P**-property of $I - S$ is proved for the Lorentz cone and some partial results were presented in the general case. Based on these results, we formulate the following:

Conjecture. *Suppose K is a symmetric cone in a Euclidean Jordan algebra V and let $S : V \to V$ be linear with $S(K) \subseteq K$ and $\rho(S) < 1$, Then $I - S$ has the* **P***-property on V.*

Acknowledgments

We thank the referees for the comments and suggestions.

References

[1] R. Balaji, On semidefinite linear complementarity problems, Ph.D. Thesis, Indian Institute of Technology—Madras, Chennai, India, December 2004.
[2] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, 1994.
[3] A. Brøndsted, An Introduction to Convex Polytopes, Springer, Berlin, 1983.
[4] R.W. Cottle, J.-S. Pang, R.E. Stone, The Linear Complementarity Problem, Academic Press, Boston, 1992.
[5] J. Faraut, A. Korányi, Analysis on Symmetric Cones, Oxford University Press, Oxford, 1994.
[6] F. Facchinei, J.-S. Pang, Finite-dimensional Variational Inequalities and Complementarity Problems, Springer-Verlag, New York, 2003.
[7] M. Fiedler, V. Pták, On matrices with non-positive off-diagonal elements and positive principal minors, Czechoslovak Math. J. 12 (1962) 382–400.
[8] M.S. Gowda, T. Parthasarathy, Complementarity forms of the theorems of Lyapunov and Stein, and related results, Linear Algebra Appl. 320 (2000) 131–144.
[9] M.S. Gowda, Y. Song, On semidefinite linear complementarity problems, Math. Prog. 88 (2000) 575–587.
[10] M.S. Gowda, Y. Song, Some new results for the semidefinite linear complementarity problem, SIAM J. Matrix Anal. Appl. 24 (2002) 25–39.
[11] M.S. Gowda, R. Sznajder, Automorphism invariance of **P** and **GUS** properties of linear transformations on Euclidean Jordan algebras, Math. Oper. Res., in press.
[12] M.S. Gowda, R. Sznajder, J. Tao, Some **P**-properties for linear transformations on Euclidean Jordan algebras, Linear Algebra Appl. 393 (2004) 203–232.
[13] R.D. Hill, S.R. Waters, On the cone of positive semidefinite matrices, Linear Algebra Appl. 90 (1987) 81–88.
[14] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[15] G. Isac, S.Z. Németh, Linear complementarity problems and matrix factorization, Research Report, Department of Mathematics and Computer Science, Royal Military College of Canada, Kingston, Ont., Canada, February 2005.
[16] S. Karamardian, An existence theorem for the complementarity problem, J. Optim. Theory Appl. 19 (1976) 227–232.
[17] H. Schneider, Positive operators and an inertia theorem, Numer. Math. 7 (1965) 11–17.
[18] S.H. Schmieta, F. Alizadeh, Extension of primal-dual interior point algorithms to symmetric cones, Math. Prog. Series A 96 (2003) 409–438.
[19] P. Stein, Some general theorems on iterants, J. Res. Natl. Bur. Standards 48 (1952) 82–83.
[20] V.A. Truong, L. Tunçel, Geometry of homogeneous convex cones, duality mapping, and optimal self-concordant barriers, Math. Prog. Series A 100 (2004) 295–316.