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On the Limiting Behavior of the Trajectory of Regularized Solutions of a P_0-Complementarity Problem

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Abstract Given a continuous $\mathbf{P}_0$-function on R^n, we consider the nonlinear complementarity problem NCP(f) and the trajectory of regularized solutions $\{x(\varepsilon) : 0 < \varepsilon < \infty\}$ where $x(\varepsilon)$ is the unique solution of NCP(f_ε) with $f_\varepsilon(x) := f(x) + \varepsilon x$. Given a sequence $\{x(\varepsilon_k)\}$ with $\varepsilon_k \downarrow 0$, we discuss (i) the existence of a bounded/convergent subsequence in the affine case, (ii) a property of any subsequential limit x^*, and (iii) the convergence of the entire trajectory in the polynomial case.

Key Words $\mathbf{P}_0$-complementarity problem, trajectory, monotone, Pareto minimal, semi-algebraic

1 INTRODUCTION

Consider the nonlinear complementarity problem NCP(f) of finding a vector x in R^n such that

$$x \geq 0, \quad f(x) \geq 0 \quad \text{and} \quad \langle f(x), x \rangle = 0, \tag{1.1}$$

where $f : R^n \to R^n$ is continuous and satisfies the $\mathbf{P}_0$-property that for all $x, y \in R^n$ with $x \neq y$,

$$\max_{\{i : x_i \neq y_i\}} (x_i - y_i)[f_i(x) - f_i(y)] \geq 0.$$

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The importance of NCP and related problems is well documented in the literature, see e.g., [6].

For $\varepsilon > 0$, let $f_\varepsilon(x) := f(x) + \varepsilon x$ be the Tikhonov regularization of f. Extending earlier results of Megiddo and Kojima (Thm. 3.4, [12]), and Facchinei and Kanzow [5] (corresponding to a continuously differentiable f), Ravindran and Gowda proved (see Corollary 6, [15]) that NCP(f_ε) has a unique solution $x(\varepsilon)$, the mapping $\varepsilon \mapsto x(\varepsilon)$ is continuous on $(0, \infty)$, and when the solution set SOL(f) of NCP(f) is nonempty and bounded, $\text{dist}(x(\varepsilon), \text{SOL}(f)) \to 0$ as $\varepsilon \to 0$. The objective of this paper is to study the limiting behavior of the trajectory $\{x(\varepsilon) : 0 < \varepsilon < \infty\}$ as $\varepsilon \to 0$. Specifically, we consider the following questions:

(*i*) Given $\varepsilon_k \downarrow 0$, when does the sequence $\{x(\varepsilon_k)\}$ have a bounded/convergent subsequence (so that a limit of some subsequence is a solution of NCP(f))?

(*ii*) If $x(\varepsilon_k) \to x^*$ for some $\varepsilon_k \downarrow 0$, what can be said about x^*?

(*iii*) When does the entire trajectory converge?

When f is monotone, that is,

$$\langle f(x) - f(y), x - y \rangle \geq 0 \quad (\forall\, x, y \in R^n),$$

there is a nice and simple answer for all the questions: Either the trajectory diverges to infinity (in the norm) in which case NCP(f) has no solution or it converges to the least two-norm solution of NCP(f) [17]. (For convergence in the setting of maximal monotone operators via Yosida approximations, see Theorem 3.5.9 in [1].) The $\mathbf{P}_0$ situation is not so simple. Even in the case of affine f given by $f(x) = Mx + q$, the NCP(f) (which now becomes the linear complementarity problem LCP(M, q)) may have a solution, yet the trajectory could diverge to infinity. Regarding question (*i*), several answers are known for the LCP. When M is a $\mathbf{P}_0 \cap \mathbf{R}_0$-matrix, every sequence $\{x(\varepsilon_k)\}$ is bounded as $\varepsilon_k \downarrow 0$ [3], [7]. By considering complementary cones associated with the given sequence $\{x(\varepsilon_k)\}$, Ebiefung [4] describes sufficient conditions for the existence of a convergent subsequence. Venkateswaran [18], based on a condition (which essentially says that the problem has a unique solution) proves that for the LCP(M, q), the entire trajectory converges. Our contribution, here, is as follows. In the affine case, by considering a 'directional perturbation' we extend results of Ebiefung, in the general case (of a continuous $\mathbf{P}_0$-function f) we show that the limit x^* mentioned in question (*ii*) is a weak Pareto minimal element of the solution set of NCP(f), and that when f is a polynomial map (in particular, affine) the entire trajectory (as $\varepsilon \downarrow 0$) either diverges to infinity (in the norm) or converges to a solution of NCP(f). One interesting aspect of the proof of this last result is the applicability of a result from algebraic geometry, which, we think should be useful to researchers working in this area.

2 PRELIMINARIES

Our reference for linear complementarity problems is [3]. For the sake of completeness, we include two important definitions.

Definition 2.1 *A matrix $M \in R^{n\times n}$ is called*

1. *a* **P**-*matrix* (**P**$_0$-*matrix*) *if all its principal minors are positive (respectively, nonnegative);*
2. *an* **R**$_0$-*matrix if* SOL$(M,0) = \{0\}$, *i.e., the homogeneous* LCP *has only the trivial solution.*

We sometimes write the complementarity problem NCP(f) as

$$x \wedge f(x) = 0$$

where '$\wedge$' denotes the componentwise minimum of vectors involved. We say that a function
$g : R^n \to R^n$ is a **P**-function if for all $x, y \in R^n$ with $x \neq y$,

$$\max_{\{i: x_i \neq y_i\}} (x_i - y_i)[g_i(x) - g_i(y)] > 0.$$

Note that if f is a **P**$_0$-function, then $f + \varepsilon I$ is a **P**-function; moreover, $f(x) = Mx + q$ is a **P**$_0$-function if and only if M is a **P**$_0$-matrix.

3 EXISTENCE OF A CONVERGENT SUBSEQUENCE

In this section, we address the first question stated in the Introduction, namely, for a given sequence $\{x(\varepsilon_k)\}$ with $\varepsilon_k \downarrow 0$, when we can expect to have a bounded/convergent subsequence. For a general continuous **P**$_0$-function, the Ravindran-Gowda result mentioned in the Introduction (also, Facchinei-Kanzow [5] for C^1 **P**$_0$-function) gives an answer when the solution set of NCP(f) is nonempty and bounded. What happens when the solution set is unbounded? We do not have an answer for the general **P**$_0$ case and so we restrict our attention to affine functions. To this end, we consider $f(x) = Mx + q$ and assume that M is a **P**$_0$-matrix.

The following simple example shows what can go wrong even in this affine case.

Example 3.1 Let

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad q = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

For every $\varepsilon > 0$, SOL$(M + \varepsilon I, q) = \{(0, \frac{1}{\varepsilon})\}$, while

$$\text{SOL}(M,q) = \{(1, x_2) : x_2 \geq 0\} \cup \{(x_1, 0) : x_1 \geq 1\}.$$

The trajectory diverges to infinity, yet keeping the same distance (one) from SOL(M,q).

Throughout this section, we fix a $\mathbf{P}_0$-matrix M, vectors p and q in R^n, and consider the problem LCP$(M+\varepsilon I, q+\varepsilon p)$ for $\varepsilon > 0$. This problem can be considered as a 'directional perturbation' of the problem LCP(M,q) in the direction of (I,p); the case $p=0$ reduces to the original problem LCP(M,q) and its perturbation LCP$(M+\varepsilon I, q)$. Since $M+\varepsilon I$ is a $\mathbf{P}$-matrix, LCP$(M+\varepsilon I, q+\varepsilon p)$ has a unique solution, say, $x(\varepsilon)$. We fix a sequence $\varepsilon_k \downarrow 0$ and seek conditions under which the sequence $\{x(\varepsilon_k)\}$ will have a convergent subsequence. Our first result relies on the boundedness of SOL(M,q).

Theorem 3.1 *Assume $M \in \mathbf{P}_0$, $\{x(\varepsilon_k)\} = \mathrm{SOL}\,(M+\varepsilon_k I, q+\varepsilon_k p)$, and $\mathrm{SOL}\,(M,q) \neq \emptyset$ and bounded. Then, $\{x(\varepsilon_k)\}$ is bounded and hence some subsequence of $\{x(\varepsilon_k)\}$ converges to a solution of* LCP (M,q).

Proof. Since SOL (M,q) is nonempty and bounded, it is stable (see Theorem 7, [8] or Theorem 1, [15]), that is, for all small $\varepsilon > 0$ and some bounded open set Ω in R^n, $\{x(\varepsilon)\} = \mathrm{SOL}\,(M+\varepsilon I, q+\varepsilon p) \subseteq \Omega$. Hence, $\{x(\varepsilon_k)\}$ is bounded and some subsequence converges. ■

When SOL(M,q) is unbounded, we can describe a condition (similar to the one in [4]) for the existence of a bounded subsequence of $\{x(\varepsilon_k)\}$ in terms of complementary cones. Recall that (Def. 1.3.2, [3]) for a given $M \in R^{n\times n}$ and $\alpha \subseteq \{1,\ldots,n\}$, the matrix $C_M(\alpha) \in R^{n\times n}$ defined by

$$C_M(\alpha)_{\cdot i} = \begin{cases} -M_{\cdot i} & \text{if } i \in \alpha, \\ I_{\cdot i} & \text{if } i \notin \alpha, \end{cases}$$

is called a complementary matrix of M (or a complementary submatrix of $(-M,I)$). The associated cone, $pos\, C_M(\alpha) := C_M(\alpha)(R^n{}_+)$, is called a *complementary cone* relative to M. Note that LCP(M,q) is solvable if and only if $q \in pos\, C_M(\alpha)$ for some α.

We now fix M, p, q, and a sequence $\{\varepsilon_k\} \downarrow 0$. For each k, there is an α (depending on k) such that $q+\varepsilon_k p \in pos\,(-(M+\varepsilon I)_\alpha, I_{\bar\alpha})$ where $\bar\alpha$ is the set of indexes not in α. Since these index sets α are finite in number, we can take a subsequence k_j so that the same α works for all k_j. Writing

$$C_{M+\varepsilon_{k_j} I}(\alpha) = \left(-(M+\varepsilon_{k_j} I)_\alpha, I_{\bar\alpha}\right) = (-M_\alpha, I_{\bar\alpha}) + \varepsilon_{k_j}\,(-I_\alpha, 0_{\bar\alpha}) = A + \varepsilon_{k_j} B,$$

we see that

$$q + \varepsilon_{k_j} p \in (A + \varepsilon_{k_j} B)(R^n{}_+) \quad (\forall j = 1,2,\ldots)$$

from which we can write $q+\varepsilon_{k_j} p = (A+\varepsilon_{k_j} B)(u(\varepsilon_{k_j}))$ where $u(\varepsilon_{k_j}) \geq 0$. It follows from a standard normalization argument that the sequence $\{u(\varepsilon_{k_j})\}$ is bounded if the following condition holds:

$$Au = 0,\ u \geq 0 \Longrightarrow u = 0\,. \tag{3.1}$$

Since $x(\varepsilon_{k_j})$ is identical to $u(\varepsilon_{k_j})$ in α components and zero in $\bar\alpha$ components, we see that (3.1) is a sufficient condition for the boundedness of $\{x(\varepsilon_{k_j})\}$.

We end this section with the monotone case. The result below and its proof are similar to the ones for the standard case (with $p = 0$) [17]. For the sake of completeness, we include a proof.

Theorem 3.2 *Let M be positive semidefinite, $p, q \in R^n$ and $\mathrm{SOL}(M,q) \neq \emptyset$. Let $\{x(\varepsilon)\} = \mathrm{SOL}(M + \varepsilon I, q + \varepsilon p)$. Then $\{x(\varepsilon)\}$ converges as $\varepsilon \downarrow 0$, to the (unique) element x^* in $\mathrm{SOL}(M,q)$ that is closest to $-p$ in the two-norm.*

Proof. Let $y(\varepsilon) = Mx(\varepsilon) + \varepsilon x(\varepsilon) + \varepsilon p + q$ and $v = Mu + q$ where $u \in \mathrm{SOL}(M,q)$. Then

$$0 \geq \langle y(\varepsilon)-v, x(\varepsilon)-u\rangle = \langle M(x(\varepsilon)-u)+\varepsilon(x(\varepsilon)+p), x(\varepsilon)-u\rangle \geq \varepsilon\langle x(\varepsilon)+p, x(\varepsilon)-u\rangle,$$

in view of the assumption that the matrix M is positive semidefinite. The above inequality shows that $\langle x(\varepsilon) + p, x(\varepsilon) - u\rangle \leq 0$. It follows that $\langle x(\varepsilon) + p, x(\varepsilon) + p - [u + p]\rangle \leq 0$. The Cauchy-Schwarz inequality gives $||x(\varepsilon) + p||^2 \leq \langle x(\varepsilon) + p, u + p\rangle \leq ||x(\varepsilon) + p|| \cdot ||u + p||$. Thus,

$$||x(\varepsilon) + p|| \leq ||u + p||. \tag{3.2}$$

This proves the boundedness of $\{x(\varepsilon) : \varepsilon > 0\}$. To prove the convergence of the entire trajectory, let $x(\varepsilon_k) \to x^*$ for some $\varepsilon_k \to 0$. By continuity and complementarity, we get $x^* \in \mathrm{SOL}(M,q)$. From (3.2), we see that for all $u \in \mathrm{SOL}(M,q)$, $||x^* + p|| \leq ||u + p||$. Since $\mathrm{SOL}(M,q)$ is convex, x^* is the (unique) element of $\mathrm{SOL}(M,q)$ that is closest to $-p$. Since the sequence $\{\varepsilon_k\}$ is arbitrary, the entire trajectory must go to x^*. This completes the proof. ■

4 WEAK PARETO MINIMAL PROPERTY

We now consider the second question raised in the Introduction: Given that $x(\varepsilon_k) \to x^*$, what is the nature of x^*? The following example shows that unlike the monotone case, in the $\mathbf{P}_0$ case, the limit x^* need not be the least two-norm solution of NCP(f).

Example 4.1 Let

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbf{P}_0 \text{ and } q = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Then

$$\mathrm{SOL}(M,q) = \{(x_1, 1) : x_1 \geq 0\},$$

$$\mathrm{SOL}(M + \varepsilon I, q) = \left\{\left(\frac{1}{1+\varepsilon}, \frac{1}{1+\varepsilon}\right)\right\},$$

and $x^* = (1,1)$.

For our next result, we need the following definition.

Definition 4.1 *Consider an element x^* of a nonempty set S. We say that x^* is a weak Pareto minimal element (Pareto minimal element) of S (with respect to the nonnegative orthant) if*

$$(x^* - int\, R^n_+) \cap S = \emptyset$$

(respectively, $(x^* - R^n_+) \cap S = \{x^*\}$). In other words, x^* is a weak Pareto minimal element of S if there is no element of S satisfying the inequality $s < x^*$ and is a Pareto minimal element of S if x^* is the only element of S satisfying the inequality $s \leq x^*$.

The above two notions appear in multi-objective optimization [11] and can be defined with respect to any cone/order.

Theorem 4.1 *Let f be a continuous $\mathbf{P}_0$-function and $\{x(\varepsilon)\} = \mathrm{SOL}(f_\varepsilon)$ Then every accumulation point of $\{x(\varepsilon)\}$ (as $\varepsilon \to 0$) is a weak Pareto minimal element of $\mathrm{SOL}(f)$.*

Proof. Suppose $x(\varepsilon_k) \to x^*$ as $\varepsilon_k \downarrow 0$. Suppose, if possible, that there exists $\bar{x} \in \mathrm{SOL}(f)$ such that $\bar{x} < x^*$. Since $0 \leq \bar{x} < x^* \in \mathrm{SOL}(f)$, by complementarity, $f(x^*) = 0$. Also, $x(\varepsilon_k) \to x^* > \bar{x} \geq 0$, so, for large k $x(\varepsilon_k) > \bar{x} \geq 0$. Again, the complementarity condition implies

$$f(x(\varepsilon_k)) + \varepsilon_k x(\varepsilon_k) = 0. \tag{4.1}$$

Since $f(x) + \varepsilon_k x$ is a **P**-function and $x(\varepsilon_k) \neq \bar{x}$ (for large k), there exists $i \in \{1, \ldots, n\}$ (depending on k) such that

$$[x(\varepsilon_k) - \bar{x}]_i [f(x(\varepsilon_k)) + \varepsilon_k x(\varepsilon_k) - (f(\bar{x}) + \varepsilon_k \bar{x})]_i > 0. \tag{4.2}$$

By taking a subsequence, we may assume that the same i works for all $k \in N$ Letting $\varepsilon_k \downarrow 0$ in (4.2), we get $[x^* - \bar{x}]_i [f(x^*) - f(\bar{x})]_i \geq 0$. By expanding this inequality, and by using the complementarity conditions $x^* \wedge f(x^*) = 0 = \bar{x} \wedge f(\bar{x})$, we get the cross-complementarity conditions

$$x^*_i (f(\bar{x}))_i = 0 = \bar{x}_i (f(x^*))_i. \tag{4.3}$$

As $x^*_i > 0$, (4.3) implies $(f(\bar{x}))_i = 0$. By (4.1) and (4.2), $[x(\varepsilon_k) - \bar{x}]_i [-\varepsilon_k \bar{x}]_i > 0$, so $[x(\varepsilon_k) - \bar{x}]_i < 0$, which contradicts the earlier condition $x(\varepsilon_k) > \bar{x}$. This proves that x^* is a weak Pareto minimal element of $\mathrm{SOL}(f)$. ∎

Remark Referring to Example 4.1, we notice that the trajectory converges to $x^* = (1, 1)$ and that x^* is a weak Pareto minimal element of the solution set. Unfortunately, every element of the solution set is a weak Pareto minimal element. This raises the following question: How does one identify x^* among all weak Pareto minimal elements of the solution set?

5 LIMITING BEHAVIOR OF THE ENTIRE TRAJECTORY

In this section, we consider the question of convergence of the entire trajectory. We restrict our attention to polynomial functions (these are functions from R^n to itself whose component functions are polynomials in n real variables) and appeal to a result from algebraic geometry. While proving the convergence of an 'interior point' trajectory in the context of monotone LCPs, Kojima, Megiddo, and Noma [10] use triangulation techniques and refer to a Referee's remark that such convergence can be established via a result from algebraic geometry. It is this algebraic geometry result that we use here. First a definition.

Definition 5.1 *A subset of R^n is called semi-algebraic if it is a finite union of sets of the form*

$$E = \{x \in R^n : p_j(x) \ \diamond_j \ 0, \ j = 1, \ldots, L\}$$

where for each j, $p_j : R^n \to R$ is a polynomial and $\diamond_j \in \{=, \leq, <\}$.

The following theorem emphasizes the key property of semi-algebraic sets needed in this section.

Theorem 5.1 *(Thm. 2.2.1, [2]) Every semi-algebraic set in R^n has a finite number of (connected) components.*

Theorem 5.2 *Consider a polynomial $\mathbf{P}_0$-function f. Let* SOL$(f_\varepsilon) = \{x(\varepsilon)\}$*, for $\varepsilon > 0$. Then the following alternative holds:*
As $\varepsilon \downarrow 0$, the trajectory $\{x(\varepsilon) : \varepsilon > 0\}$ either converges to an element of SOL(f) *or diverges to infinity in the norm.*

Proof. Suppose that the trajectory $\{x(\varepsilon) : \varepsilon > 0\}$ does not diverge to infinity in the norm, in which case we may assume that for some sequence $\varepsilon_k \downarrow 0$, $x(\varepsilon_k) \to x^*$. Clearly, $x^* \in$ SOL(f). We claim that $x(\varepsilon) \to x^*$ as $\varepsilon \to 0$. For any $\delta > 0$, let $B_\delta := \{x : -\delta \leq x_i - x_i^* \leq \delta, \ i = 1, 2, \ldots, n\}$ be a box around x^*. To prove the convergence $x(\varepsilon) \to x^*$, we show that for any such δ, there exists $\bar{\varepsilon} > 0$ such that for $0 < \varepsilon < \bar{\varepsilon}$, $x(\varepsilon) \in B_\delta$. Assume the contrary that for some $\delta > 0$ there exists a sequence $\{\bar{\varepsilon}_l\} \downarrow 0$ such that $x(\bar{\varepsilon}_l) \notin B_\delta$. By renaming the sequences, if necessary, we may assume that

(a) $x(\varepsilon_k) \in B_\delta$, $x(\bar{\varepsilon}_k) \notin B_\delta$ for all k and

(b) $1 > \varepsilon_1 > \bar{\varepsilon}_1 > \varepsilon_2 > \bar{\varepsilon}_2 > \varepsilon_3 > \cdots$.

Now consider the set

$$E := \{(x, \varepsilon) : \varepsilon > 0, \ x \geq 0, \ f(x) + \varepsilon x \geq 0, \ x_i(f_i(x) + \varepsilon x_i) = 0 \text{ for } i = 1, 2, \ldots, n\} \tag{5.1}$$

which is really the graph $\{(x(\varepsilon), \varepsilon) : \varepsilon > 0\}$. Since the function f is a polynomial, the sets E and $E \cap B_\delta \times (0, 1)$ are semi-algebraic.

We show that the semi-algebraic set $E \cap B_\delta \times (0,1)$ has infinitely many cor ponents, thus reaching a contradiction to Theorem 5.1. For each $j = 1, 2, ..$ let C_j be the (connected) component of the set $E \cap B_\delta \times (0,1)$ containin $(x(\varepsilon_j), \varepsilon_j)$. Since connected sets in R are intervals, C_j must be of the forr $\{(x(\varepsilon), \varepsilon) : \varepsilon \in I_j\}$ where I_j is an interval in $(0,1)$ containing ε_j. Sinc $x(\bar{\varepsilon}_l) \notin B_\delta$ for all l, we see that no $\bar{\varepsilon}_l$ can be in any I_j. By the interlacin property (b) above, the intervals I_j $(j = 1, 2 \ldots)$ and hence the component C_j $(j = 1, 2, \ldots)$ are pairwise disjoint. Thus we see that there are infinitel many components in $E \cap B_\delta \times (0,1)$ contradicting Theorem 5.1. This prove that $x(\varepsilon) \to x^*$. ∎

Remarks (1) In the above proof we relied on Theorem 5.1. Instead, we coul have used the so called *curve selection theorem* (Prop. 2.6.19, [2]) which say that if X is a semi-algebraic set in R^n and x^* is an accumulation point of X then there is a continuous curve in X that begins at x^*.
(2) For the linear complementarity problem, R. Stone [16] has given an elemen tary proof of the above theorem based on complementary cones and rationa functions.

We now state our final result for linear complementarity problems withou proof.

Corollary 5.1 *Let M be a $\mathbf{P}_0$-matrix,* SOL$(M + \varepsilon I, q) = \{x(\varepsilon)\}$ *for* $\varepsilon > 0$. *Then as $\varepsilon \downarrow 0$, either the trajectory $\{x(\varepsilon) : \varepsilon > 0\}$ diverges to infinity in th norm or converges to a weak Pareto minimal point of* SOL(M, q).

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