On the Uniform Nonsingularity Property for Linear Transformations on Euclidean Jordan Algebras*

Roman Sznajder[†], M. Seetharama Gowda[‡], and Jiyuan Tao[§]

October 25, 2011

Communicated by Florian A. Potra

Abstract. In a recent paper, Chua and Yi introduced the so-called uniform nonsingularity property for a nonlinear transformation on a Euclidean Jordan algebra and showed that it implies the global uniqueness property in the context of symmetric cone complementarity problems. In a related paper, Chua, Lin and Yi raise the question of converse. In this paper, we show that, for linear transformations, the uniform nonsingularity property is inherited by principal subtransformations and, on simple algebras, it is invariant under the action of cone automorphisms. Based on these results, we answer the question of Chua, Lin and Yi in the negative.

Key Words: Uniform nonsingular property, Principal subtransformation, Euclidean Jordan algebra, Symmetric cone, Complementarity problem.

MSC2010 *subject classification:* Primary: 90C33, Secondary: 17C55, 17C20

*Thanks are due to two anonymous referees for their insightful comments and suggestions.

[†](corresponding author) Department of Mathematics, Bowie State University, Bowie, Maryland 20715, e-mail: `rsznajder@bowiestate.edu`

[‡]Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, Maryland, 21250, e-mail: `gowda@math.umbc.edu`

[§]Department of Mathematics and Statistics, Loyola University Maryland, Baltimore, Maryland 21210, e-mail: `jtao@loyola.edu`

1 Introduction

The class of real square matrices with all principal minors positive was introduced by Fiedler and Pták [1]. Such matrices have wide ranging applications, particularly in matrix theory and optimization; see e.g., [2]. It is well known that this positive principal minor property can also be described by the nonsign-reversal property as well as the uniqueness of solution in all corresponding linear complementarity problems [3].

In [4], this nonsign-reversal property was extended to linear transformations on symmetric cones in Euclidean Jordan algebras. In a recent and very interesting paper, Chua and Yi [5] showed that this extended nonsign-reversal property is equivalent to a condition described entirely in terms of the norm and spectral decompositions of elements of the Euclidean Jordan algebra. They also introduced, for nonlinear transformations, a property – called the uniform nonsingularity property – by replacing the spectral decompositions by Peirce decompositions and further showed that under this property, one obtains uniqueness of solutions in the corresponding symmetric cone nonlinear complementarity problem. In a subsequent paper [6], Chua, Lin, and Yi raise the question of converse. In this paper, we consider this uniform nonsingularity property for linear transformations and prove several results: We show that this property is inherited by all principal subtransformations, and on simple algebras it is invariant under cone automorphisms and implies global unique solvability in all related symmetric cone linear complementarity problems. We finally answer Chua, Lin and Yi's question in the negative.

The organization of the paper is as follows. In Section 2, we recall/introduce basic definitions, concepts, and notation needed in the paper. Section 3 deals with the inheritance of the uniform nonsingularity property by principal subtransformations. In Section 4, we investigate the invariance of the uniform nonsingularity property under cone automorphisms and study the ultra nonsign-reversal property as a consequence of the uniform nonsingularity property. Section 5 deals with a necessary condition for the uniform nonsingularity property on $\mathcal{S}^n$. Finally, Section 6 concludes the paper.

2 Preliminaries

2.1 Euclidean Jordan Algebras

We briefly recall some concepts and notation used in this paper. Most of these can be found in [7] and [4].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle \cdot, \cdot \rangle)$ where $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over $\mathbb{R}$ and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions for all x and y: $x \circ y = y \circ x$, $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where

$x^2 := x \circ x$, and $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$. In addition, we assume that V has an element $e \in V$–called the *unit* element–satisfying $x \circ e = x$ for all $x \in V$.

A Euclidean Jordan algebra is said to be *simple* if it is not a direct sum of two Euclidean Jordan algebras. It is well known that any Euclidean Jordan algebra is a direct sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of the following:

(i) The Jordan spin algebra $\mathcal{L}^n$ for $n > 1$.

(ii) The space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices.

(iii) The space $\mathcal{H}^n$ of all $n \times n$ complex Hermitian matrices.

(iv) The space $\mathcal{Q}^n$ of all $n \times n$ quaternionic Hermitian matrices.

(v) The space $\mathcal{O}^3$ of all 3×3 octonionic Hermitian matrices.

Throughout this paper, we let $(V, \circ, \langle \cdot, \cdot \rangle)$ denote a Euclidean Jordan algebra of rank r. The symmetric cone of V is the cone of squares $K := \{x \circ x : x \in V\}$. We use the notation $x \geq 0$ ($x > 0$) when $x \in K$ (respectively, $x \in \text{interior}(K)$).

An element $c \in V$ such that $c^2 = c$ is called an *idempotent* in V; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say a finite set $\{e_1, e_2, \ldots, e_r\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \text{ if } i \neq j, \text{ and } \sum_1^r e_i = e.$$

The spectral decomposition For any $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that $x = \lambda_1 e_1 + \cdots + \lambda_r e_r$. The numbers λ_i are called the *spectral eigenvalues* of x. We say that an element x is *invertible* in V if all its spectral eigenvalues are nonzero.

The Peirce decomposition Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = \mathbb{R}\, e_i$$

and when $i \neq j$,

$$V_{ij} := \{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\}.$$

Then we have the following

Theorem 2.1 ([7], Theorem IV.2.1) *The space V is the orthogonal direct sum of spaces V_{ij} $(i \leq j)$. Furthermore,*

$$\begin{aligned}
&V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj},\\
&V_{ij} \circ V_{jk} \subset V_{ik} \textit{ if } i \neq k, \textit{ and}\\
&V_{ij} \circ V_{kl} = \{0\} \textit{ if } \{i, j\} \cap \{k, l\} = \emptyset.
\end{aligned}$$

Thus, given a Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{1 \leq i \leq j \leq r} x_{ij},$$

where $x_{ij} \in V_{ij}$. This expression is the **Peirce decomposition** of x with respect to $\{e_1, e_2, \ldots, e_r\}$. For any given idempotent $c \in V$,

$$V(c, 1) := \{x \in V : x \circ c = x\}$$

is a subalgebra of V (Proposition IV.1.1 in [7]).

Lyapunov transformation and quadratic representation For a given $a \in V$, the *Lyapunov transformation* L_a and *quadratic representation* $P_a : V \to V$ are defined, respectively, by

$$L_a(x) = a \circ x \qquad \text{and} \qquad P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

We say that two elements x and y *operator commute* if $L_x L_y = L_y L_x$. It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame (Lemma X.2.2 in [7]).

Principal subtransformation Given a linear transformation $L : V \to V$ and an idempotent $c \in V$, the transformation $\widehat{L} : V(c, 1) \to V(c, 1)$, given by

$$\widehat{L}(x) = \Pi_{V(c,1)} L(x) = P_c L(x),$$

is called a *principal subtransformation* of L. Here $\Pi_{V(c,1)} = P_c$ represents the orthogonal projection mapping onto $V(c, 1)$ ([7], p.64). The determinant of $\widehat{L}$ is called a *principal minor* of L. We say that L has the *principal minor property* if every principal minor of L is positive.

Automorphisms A linear transformation $\Lambda : V \to V$ is said to be an *algebra automorphism* if Λ is invertible and $\Lambda(x \circ y) = \Lambda(x) \circ \Lambda(y)$ for all $x, y \in V$. A linear transformation $\Gamma : V \to V$ is said to be a *cone automorphism* if $\Gamma(K) = K$. The set of all automorphisms of V (of K) is denoted by $\mathrm{Aut}(V)$ (respectively, $\mathrm{Aut}(K)$).

2.2 P, UNS, and Complementarity Properties

Throughout this paper, we assume that $L : V \to V$ be a linear transformation. Given L on V and $q \in V$, the linear complementarity problem, $\mathrm{LCP}(L, K, q)$, is to find an $x \in V$ such that

$$x \in K, \; y = L(x) + q \in K, \text{ and } \langle x, y \rangle = 0.$$

Definition 2.1 Given L on V, we say that L has/is

(a) the *monotonicity (strict) property* iff $\langle L(x), x\rangle \geq 0$ (respectively, > 0) for all $0 \neq x \in V$;

(b) the *uniform nonsingularity property* (**UNS**-*property*) iff there exists $\alpha > 0$ such that for any Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V, any $x \in V$ with its Peirce decomposition $x = \sum_{i\leq j} x_{ij}$ (with respect to the given Jordan frame), and any nonnegative matrix $D = [d_{ij}] \in \mathcal{S}^r$,

$$\|L(x) + D \bullet x\| \geq \alpha\|x\|, \tag{1}$$

where, $D \bullet x := \sum_{i\leq j} d_{ij}x_{ij}$;

(c) the **P**-*property* iff

$$\left.\begin{array}{r} x \text{ and } L(x) \text{ operator commute} \\ x \circ L(x) \leq 0 \end{array}\right\} \Rightarrow x = 0; \tag{2}$$

Note: This is the nonsign-reversal property mentioned in the Introduction.

(d) the *ultra* **P**-*property* iff for any $\Gamma \in \mathrm{Aut}(K)$, every principal subtransformation of $\widehat{L} := \Gamma^T L\Gamma$ has the **P**-property;

(e) the *globally uniquely solvable (***GUS***)-property* iff $\mathrm{LCP}(L, K, q)$ has a unique solution for all $q \in V$;

(g) a *Lyapunov-like transformation* iff

$$x, y \in K \text{ and } \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0.$$

Remark 2.1 The **P**-property is a generalization of the standard **P**-matrix property, see [3]. The **P**-property on $\mathcal{S}^n$ first appeared in [8] in connection with Lyapunov transformations; our definition given above comes from [4]. For the **P**-property on $\mathcal{L}^n$, see Tao's dissertation [9]. In [5], Chua and Yi show that the above **P**-property is equivalent to the following norm condition: There exists an $\alpha > 0$ such that for every $x \in V$ with spectral decomposition $x = \sum_{i=1}^{r} x_i e_i$ and for all $d_i \geq 0$ (in $\mathbb{R}$),

$$\|L(x) + \sum_{i=1}^{r} d_i x_i e_i\| \geq \alpha\|x\|.$$

Specialized to the algebra $\mathbb{R}^n$ (with the componentwise product as the Jordan product and the usual inner product), this result gives a new characterization of **P**-matrices in terms of the norm: M is a **P**-matrix if and only if there is an $\alpha > 0$ such that for all $x \in \mathbb{R}^n$ and for all nonnegative diagonal matrices D in $\mathbb{R}^{n\times n}$,

$$\|Mx + Dx\| \geq \alpha\|x\| \quad \forall\, x \in \mathbb{R}^n. \tag{3}$$

Remark 2.2 (1) The ultra **P**-property first appeared in the form of so-called $\mathbf{P}_2$-property in $\mathcal{S}^n$, see [8], and later in [10], once again for $\mathcal{S}^n$. For the present formulation and related properties, see [11].
(2) The **GUS**-property is a generalization of the **P**-property for matrices on R^n, see [4].
(3) Lyapunov-like transformations are particular instances of the so-called **Z**-transformations, see [12] for details.

3 Inheritance of the UNS-Property by Principal Subtransformations

Proposition 3.1 *The linear transformation L on V has the* **UNS***-property if and only if there exists an $\alpha > 0$ such that for any Jordan frame $\{e_1, e_2, \dots, e_r\}$ in V, any $x \in V$,*

$$\sum_{\{(i,j):\langle x_{ij}, y_{ij}\rangle \geq 0,\, x_{ij}\neq 0\}} \|y_{ij}\|^2 + \sum_{\{(i,j):\langle x_{ij}, y_{ij}\rangle < 0\}} \frac{\|x_{ij}\|^2\|y_{ij}\|^2 - \langle x_{ij}, y_{ij}\rangle^2}{\|x_{ij}\|^2} \geq \alpha^2\|x\|^2, \qquad (4)$$

where $x = \sum_{i\leq j} x_{ij}$ and $y := L(x) = \sum_{i\leq j} y_{ij}$ are the Peirce decompositions with respect to $\{e_1, e_2, \dots, e_r\}$.

Proof. Assume that L have the **UNS**-property. Fix a Jordan frame $\{e_1, e_2, \dots, e_r\}$, element x, and $y = L(x)$ with their corresponding Peirce decompositions. Using the orthogonality of the Peirce spaces, the inequality (1) reads:

$$\sum_{i\leq j} \|y_{ij} + d_{ij}x_{ij}\|^2 \geq \alpha^2\|x\|^2 \text{ for all } d_{ij} \geq 0. \qquad (5)$$

Expanding a typical term on the left-hand side of the above inequality, we get

$$\|y_{ij} + d_{ij}x_{ij}\|^2 = \|y_{ij}\|^2 + 2d_{ij}\langle x_{ij}, y_{ij}\rangle + \|x_{ij}\|^2 d_{ij}^2.$$

Using elementary algebra, we obtain

$$\min_{d_{ij}\geq 0} \|y_{ij} + d_{ij}x_{ij}\|^2 = \begin{cases} \|y_{ij}\|^2 & \text{if } \|x_{ij}\|^2 = 0, \\ \frac{\|x_{ij}\|^2\|y_{ij}\|^2 - \langle x_{ij}, y_{ij}\rangle^2}{\|x_{ij}\|^2} & \text{if } \|x_{ij}\|^2 \neq 0 \text{ and } \langle x_{ij}, y_{ij}\rangle < 0, \\ \|y_{ij}\|^2 & \text{if } \|x_{ij}\|^2 \neq 0 \text{ and } \langle x_{ij}, y_{ij}\rangle \geq 0. \end{cases}$$

As a consequence, (5) is equivalent to:

$$\sum_{\{(i,j):\langle x_{ij}, y_{ij}\rangle \geq 0\}} \|y_{ij}\|^2 + \sum_{\{(i,j):\langle x_{ij}, y_{ij}\rangle < 0\}} \frac{\|x_{ij}\|^2\|y_{ij}\|^2 - \langle x_{ij}, y_{ij}\rangle^2}{\|x_{ij}\|^2} \geq \alpha^2\|x\|^2. \qquad (6)$$

We will show that this inequality reduces to (4) by proving that the terms in the first summation in (6) corresponding to the indices with $x_{ij} = 0$ can be dropped. To this end, fix a pair (i, j) with $x_{ij} = 0$. With $u = L(y)$ and $\varepsilon > 0$, we define the following sets within $\{(i, j) : 1 \le i \le j \le r\}$:

$$
\begin{aligned}
I_\varepsilon = \{(i,j) : \langle (x - \varepsilon y)_{ij}, (y - \varepsilon u)_{ij} \rangle \ge 0\},\\
\mathcal{A} = I_\varepsilon \cap \{(i,j) : x_{ij} = 0\},\\
\mathcal{B} = I_\varepsilon \cap \{(i,j) : x_{ij} \neq 0\},\\
\mathcal{C} = I_\varepsilon^c \cap \{(i,j) : x_{ij} = 0\},\\
\mathcal{D} = I_\varepsilon^c \cap \{(i,j) : x_{ij} \neq 0\},
\end{aligned}
$$

where $I_\varepsilon^c = \{(i,j) : \langle (x - \varepsilon y)_{ij}, (y - \varepsilon u)_{ij} \rangle < 0\}$. As these index sets are finite, we may consider a sequence $\varepsilon_k \downarrow 0$ such that all I_{ε_k} are identical. For notational convenience, we write $\varepsilon \to 0$ to mean $\varepsilon_k \downarrow 0$. As α in the inequality (6) is independent of x and y, we may replace x by $x - \varepsilon y$ and y by $L(x - \varepsilon y) = y - \varepsilon u$ in (6) to get

$$
\begin{aligned}
&\sum_{\mathcal{A}} \|(y - \varepsilon u)_{ij}\|^2 + \sum_{\mathcal{B}} \|(y - \varepsilon u)_{ij}\|^2\\
&+ \sum_{\mathcal{C}} \frac{\|(x - \varepsilon y)_{ij}\|^2 \|(y - \varepsilon u)_{ij}\|^2 - \langle (x - \varepsilon y)_{ij}, (y - \varepsilon u)_{ij} \rangle^2}{\|x_{ij} - \varepsilon y_{ij}\|^2}\\
&+ \sum_{\mathcal{D}} \frac{\|(x - \varepsilon y)_{ij}\|^2 \|(y - \varepsilon u)_{ij}\|^2 - \langle (x - \varepsilon y)_{ij}, (y - \varepsilon u)_{ij} \rangle^2}{\|x_{ij} - \varepsilon y_{ij}\|^2} \ge \alpha^2 \|x - \varepsilon y\|^2.
\end{aligned}
\tag{7}
$$

Now we consider the behavior of each of the above terms when $\varepsilon \to 0$.

For $(i, j) \in \mathcal{A}$, $\langle x_{ij} - \varepsilon y_{ij}, y_{ij} - \varepsilon u_{ij} \rangle \ge 0$. Since $x_{ij} = 0$, we have $-\|y_{ij}\|^2 + \varepsilon \langle y_{ij}, u_{ij} \rangle \ge 0$, hence by letting $\varepsilon \to 0$, $y_{ij} = 0$.

For $(i, j) \in \mathcal{C}$, $\langle x_{ij} - \varepsilon y_{ij}, y_{ij} - \varepsilon u_{ij} \rangle < 0$, and so $y_{ij} \neq 0$; Whence

$$
\begin{aligned}
&\frac{\|0 - \varepsilon y_{ij}\|^2 \|y_{ij} - \varepsilon u_{ij}\|^2 - \langle 0 - \varepsilon y_{ij}, y_{ij} - \varepsilon u_{ij} \rangle^2}{\varepsilon^2 \|y_{ij}\|^2}\\
=&\|y_{ij} - \varepsilon u_{ij}\|^2 - \frac{\langle y_{ij}, y_{ij} - \varepsilon u_{ij} \rangle^2}{\|y_{ij}\|^2} \to \|y_{ij}\|^2 - \|y_{ij}\|^2 = 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{\mathcal{A}} \|(y - \varepsilon u)_{ij}\|^2 \to \sum_{\mathcal{A}} \|y_{ij}\|^2 = 0, \ \sum_{\mathcal{B}} \|(y - \varepsilon u)_{ij}\|^2 \to \sum_{\mathcal{B}} \|y_{ij}\|^2,\\
&\sum_{\mathcal{C}} \frac{\|(x - \varepsilon y)_{ij}\|^2 \|(y - \varepsilon u)_{ij}\|^2 - \langle (x - \varepsilon y)_{ij}, (y - \varepsilon u)_{ij} \rangle^2}{\|x_{ij} - \varepsilon y_{ij}\|^2} \to 0,
\end{aligned}
$$

and

$$\sum_{\mathcal{D}} \frac{\|(x-\varepsilon y)_{ij}\|^2\|(y-\varepsilon u)_{ij}\|^2 - \langle (x-\varepsilon y)_{ij}, (y-\varepsilon u)_{ij}\rangle^2}{\|x_{ij}-\varepsilon y_{ij}\|^2}$$
$$\to \sum_{\mathcal{D}} \frac{\|x_{ij}\|^2\|y_{ij}\|^2 - \langle x_{ij}, y_{ij}\rangle^2}{\|x_{ij}\|^2}.$$

In summary, when $\varepsilon \to 0$, the inequality (7) becomes

$$\sum_{\mathcal{B}} \|y_{ij}\|^2 + \sum_{\mathcal{D}} \frac{\|x_{ij}\|^2\|y_{ij}\|^2 - \langle x_{ij}, y_{ij}\rangle^2}{\|x_{ij}\|^2} \geq \alpha^2\|x\|^2. \tag{8}$$

Observe that as $\varepsilon \to 0$, $(i,j) \in \mathcal{B}$ implies $\langle x_{ij}, y_{ij}\rangle \geq 0$ and $x_{ij} \neq 0$.
Similarly, $(i,j) \in \mathcal{D}$ implies $\langle x_{ij}, y_{ij}\rangle \leq 0$ and $x_{ij} \neq 0$. Now, we have $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where $\mathcal{D}_1 := \{(i,j) \in \mathcal{D} : \langle x_{ij}, y_{ij}\rangle < 0\}$ and $\mathcal{D}_2 = \{(i,j) \in \mathcal{D} : \langle x_{ij}, y_{ij}\rangle = 0, x_{ij} \neq 0\}$. Thus, (8) becomes

$$\sum_{\mathcal{B}} \|y_{ij}\|^2 + \sum_{\mathcal{D}_1} \frac{\|x_{ij}\|^2\|y_{ij}\|^2 - \langle x_{ij}, y_{ij}\rangle^2}{\|x_{ij}\|^2} + \sum_{\mathcal{D}_2} \|y_{ij}\|^2 \geq \alpha^2\|x\|^2.$$

Hence,

$$\sum_{\mathcal{B}\cup\mathcal{D}_2} \|y_{ij}\|^2 + \sum_{\mathcal{D}_1} \frac{\|x_{ij}\|^2\|y_{ij}\|^2 - \langle x_{ij}, y_{ij}\rangle^2}{\|x_{ij}\|^2} \geq \alpha^2\|x\|^2.$$

Now,

$$\mathcal{B} \cup \mathcal{D}_2 \subseteq \{(i,j) : \langle x_{ij}, y_{ij}\rangle \geq 0, x_{ij} \neq 0\} \text{ and } \mathcal{D}_1 \subseteq \{(i,j) : \langle x_{ij}, y_{ij}\rangle < 0\}.$$

Thus, the above inequality implies

$$\sum_{\{(i,j):\langle x_{ij}, y_{ij}\rangle \geq 0,\, x_{ij}\neq 0\}} \|y_{ij}\|^2 + \sum_{\{(i,j):\langle x_{ij}, y_{ij}\rangle < 0\}} \frac{\|x_{ij}\|^2\|y_{ij}\|^2 - \langle x_{ij}, y_{ij}\rangle^2}{\|x_{ij}\|^2} \geq \alpha^2\|x\|^2.$$

This proves that the **UNS**-property implies (4). Since the inequality (4) implies (6), the converse implication is obvious. □

Remark 3.1 We note that (6) and (4) are similar except that in the first summation of (4), we consider only those y_{ij} for which the corresponding $x_{ij} \neq 0$.

Corollary 3.1 ([5], Proposition 4.4) *Suppose $L : V \to V$ have the* **UNS***-property. Then L has the* **P***-property.*

This can be seen as follows: Let x and $y = L(x)$ operator commute and $x \circ L(x) \leq 0$. Write the spectral decompositions $x = \sum_1^r x_i e_i$ and $y = \sum_1^r y_i e_i$ (so that $x_{ii} = x_i e_i$, etc.). When $x \neq 0$, using (4), we get

$$\sum_{\{i:\, x_i y_i \|e_i\|^2 \geq 0,\, x_i \neq 0\}} \|y_{ii}\|^2 \geq \alpha^2\|x\|^2,$$

which implies $y_i \neq 0$ for some i with $x_i y_i \|e_i\|^2 \geq 0$ and $x_i \neq 0$. Thus, $x_i y_i \|e_i\|^2 > 0$, which contradicts the fact that $x \circ y \leq 0$. Hence, $x = 0$.

Theorem 3.1 *Suppose that L have the* **UNS**-*property. Then every principal subtransformation of L also has the* **UNS**-*property.*

Proof. For any idempotent $c \in V$, we consider the subalgebra $V(c,1)$. Recall that the principal subtransformation of L, determined by c and defined on $V(c,1)$, is given by

$$\widehat{L}(x) = \Pi_{V(c,1)} L(x) = P_c L(x),$$

where $\Pi_{V(c,1)}$ represents the orthogonal projection onto $V(c,1)$.

Given any Jordan frame $\{e_1, e_2, \ldots, e_k\}$ in $V(c,1)$, we extend it to a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V. For any $x \in V(c,1)$ and $y = L(x)$, we write the Peirce decompositions with respect to $\{e_1, e_2, \ldots, e_r\}$:

$$x = \sum_{1 \leq i \leq j \leq r} x_{ij} \quad \text{and} \quad y = \sum_{1 \leq i \leq j \leq r} y_{ij}.$$

By Lemma 20 in [4], $x_{ij} = 0$ for any index (i,j) outside of the set $\{(i,j) : 1 \leq i \leq j \leq k\}$. We also have $\widehat{y} := \widehat{L}(x) = P_c(y) = \sum_{1 \leq i \leq j \leq k} y_{ij}$. Since L has the **UNS**-property, Proposition 3.1 indicates that in (4) we can ignore terms y_{ij} with indices (i,j) outside of the set $\{(i,j) : 1 \leq i \leq j \leq k\}$. This means that we have an inequality of type (4) now for $\widehat{L}$ on $V(c,1)$. This precisely means that $\widehat{L}$ has the **UNS**-property. □

Corollary 3.2 *For any linear transformation L on V, consider the following statements:*

(a) L has the **UNS**-*property.*

(b) All principal subtransformations of L have the **P**-*property.*

(c) L has the positive principal minor property.

Then $(a) \Rightarrow (b) \Rightarrow (c)$.

Proof. The implication $(a) \Rightarrow (b)$ follows from Theorem 3.1 and Corollary 3.1.
The implication $(b) \Rightarrow (c)$ follows from the fact that every **P**-transformation has positive determinant, see Theorem 11 in [4]. □

4 Cone Invariance and the Ultra P-Property of UNS-Transformations

Theorem 4.1 *Suppose V be simple and L have the* **UNS**-*property. Then for any $\Gamma \in Aut(K)$, $\Gamma^T L\Gamma$ also has the* **UNS**-*property.*

Proof. First, we show that $\forall\, \Lambda \in \mathrm{Aut}(V)$, $\Lambda^T L\Lambda$ has the **UNS**-property. Towards this, fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$, an element x with the corresponding Peirce decomposition $x = \sum x_{ij}$, and $d_{ij} \geq 0$ in $\mathbb{R}$.
Fix $\Lambda \in \mathrm{Aut}(V)$. Then $\{\Lambda(e_1), \Lambda(e_2), \ldots, \Lambda(e_r)\}$ is a Jordan frame in V and $\Lambda(x) = \sum \Lambda(x_{ij})$ is the corresponding Peirce decomposition of $\Lambda(x)$. Now,

$$\begin{aligned} & \|\Lambda^T L\Lambda(x) + \sum d_{ij}x_{ij}\| = \|\Lambda^T(L(\Lambda(x))) + \sum d_{ij}x_{ij}\| \\ = \; & \|\Lambda^{-1}[L(\Lambda(x)) + \sum d_{ij}\Lambda(x_{ij})]\| = \|L(\Lambda(x)) + \sum d_{ij}\Lambda(x_{ij})\| \\ \geq \; & \alpha\|\Lambda(x)\| = \alpha\|x\|. \end{aligned}$$

Here, the third and the last equalities are due to the fact that in a simple algebra V, every $\Lambda \in \mathrm{Aut}(V)$ is an orthogonal transformation (see [7], p.56), and the inequality holds because of the **UNS**-property of L. This proves that $\Lambda^T L\Lambda$ has the **UNS**-property.

Next, we show that for every a invertible in V, $P_a L P_a$ has the **UNS**-property.
We consider two cases: First assume that the spectral decomposition of a is given by $a = \sum_{i=1}^r a_i e_i$. Then (see [13])

$$P_a(x_{ij}) = a_i a_j x_{ij} \quad \text{and} \quad P_a^{-1}(x_{ij}) = \frac{x_{ij}}{a_i a_j}.$$

Let $z := P_a(x)$ and $\|P_a^{-1}\| = \dfrac{1}{c}$. Then,

$$\begin{aligned} & \|(P_a L P_a)(x) + \sum d_{ij}x_{ij}\| \\ = \; & \|P_a[L(P_a(x)) + P_a^{-1}(\sum d_{ij}x_{ij})]\| \geq c\|L(P_a(x)) + P_a^{-1}(\sum d_{ij}x_{ij})\| \\ = \; & c\left\|L(z) + \sum \frac{d_{ij}}{a_i a_j} x_{ij}\right\| = c\left\|L(z) + \sum \frac{d_{ij}}{(a_i a_j)^2} z_{ij}\right\| \\ \geq \; & c\alpha\|z\| = c\alpha\|P_a(x)\| \geq c^2\alpha\|x\|. \end{aligned}$$

Now suppose that the spectral decomposition of a be given by $a = \sum_{i=1}^r a_i f_i$, where $\{f_1, f_2, \ldots, f_r\}$ is some Jordan frame in V. Then, as V is simple, there exists a $\Lambda \in \mathrm{Aut}(V)$ such that $f_i = \Lambda(e_i)$ for all i. Let $b := \sum a_i e_i$, so $a = \Lambda(b)$. Observe that

$$P_a = P_{\Lambda(b)} = \Lambda P_b \Lambda^{-1}.$$

Now, by what has been proved earlier, $\widehat{L} := \Lambda^{-1}L\Lambda = \Lambda^T L\Lambda$ has the **UNS**-property (with no change in the constant α). In addition, by the above case,

$$\|(P_b\widehat{L}P_b)(x) + \sum d_{ij}x_{ij}\| \geq c^2\alpha||x||,$$

where now $||P_b^{-1}|| = \frac{1}{c}$. Thus, we have,

$$\begin{aligned} & \|P_aLP_a(x) + \sum d_{ij}x_{ij}\| = \|\Lambda P_b\Lambda^{-1}L\Lambda P_b\Lambda^{-1}(x) + \sum d_{ij}x_{ij}\| \\ = & \ \|(\Lambda\widehat{\widehat{L}}\Lambda^{-1})(x) + \sum d_{ij}x_{ij}\| \geq \beta\|x\|, \end{aligned}$$

where the constant β depends on α and the norm of P_b^{-1}. Here, $\widehat{\widehat{L}} = P_b(\Lambda^{-1}L\Lambda)P_b$ and the last inequality comes from the inequalities proved earlier for algebra automorphisms.

Finally, consider $\Gamma \in \text{Aut}(K)$. As V is simple, we may write $\Gamma = P_a\Lambda$ for some $\Lambda \in \text{Aut}(V)$ and $a > 0$, see [7], p.56. Now, $\Gamma^T L\Gamma = \Lambda^T(P_aLP_a)\Lambda$. By the above special cases, $\Gamma^T L\Gamma$ also has the **UNS**-property. □

Theorem 4.2 *Suppose that V be simple and L have the* **UNS***-property. Then L has ultra* **P** *and* **GUS** *properties.*

Proof. Consider any $\Gamma \in \text{Aut}(K)$. By the previous theorem, $\Gamma^T L\Gamma$ has the **UNS**-property. By Corollary 3.2, every principal subtransformation of $\Gamma^T L\Gamma$ has the **P**-property. This proves that L has the ultra **P**-property. Since the ultra **P**-property implies the **GUS**-property, see Theorem 6.2 in [11], we have the stated conclusion. □

The following result shows that in a number of situations, the **UNS**-property reduces to strict monotonicity.

Corollary 4.1 *L has the* **UNS***-property if and only if L is strictly monotone under each of the following conditions:*

(a) L is self-adjoint.

(b) L is Lyapunov-like.

Proof. We know that strict monotonicity implies the **UNS**-property (see Theorem 3.1 in [6]). Now assume that L has the **UNS**-property. Then it has the **P**-property and the positive principal minor property (by Corollary 3.2). In particular, all real eigenvalues are positive (see Theorem 11 in [4]) and $\langle L(c), c\rangle > 0$ for all primitive idempotents (which can be seen by considering principal subtransformations of L on $V(c, 1) = R\,c$). When L is symmetric, all eigenvalues of L are real and hence positive, leading to strict monotonicity.

When L is Lyapunov-like, for any $0 \neq x \in V$ with its spectral decomposition $x = \sum_{i=1}^{r} x_i e_i$, we have

$$\langle L(x), x\rangle = \langle L(\sum_{i=1}^{r} x_i e_i), \sum_{i=1}^{r} x_i e_i\rangle = \sum_{i=1}^{r} x_i^2 \langle L(e_i), e_i\rangle > 0.$$

Hence, L is strictly monotone. $\square$

Remark 4.1 The above result applies to the following transformations:

- L_a - the Lyapunov transformation associated with an element $a \in V$,
- P_a - the quadratic representation associated with an element $a \in V$,
- L_A on $\mathcal{S}^n$ defined by $L_A(X) := AX + XA^T$.

The first two transformations are self-adjoint, the last one is Lyapunov-like on $\mathcal{S}^n$ (see [12]).

It is known that

(i) L_a is strictly monotone if and only if $a > 0$, see, e.g., [13], Proposition 4,

(ii) P_a is strictly monotone if and only if $\pm a > 0$ when V is simple, see [12], Theorem 6.5.

It is also known (see [8]) that L_A is/has

(i) Strictly monotone if and only if A is positive definite,

(ii) the **P**-property if and only if A is positive stable, i.e., all its eigenvalues have positive real parts,

(iii) the **GUS**-property if and only if A positive stable and positive semidefinite.

Thus, according to Corollary 4.1, L_A has the **UNS**-property on $\mathcal{S}^n$ if and only if A is positive definite. Now, consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}.$$

This is positive stable and positive semidefinite, but not positive definite. Thus L_A has the **GUS**-, but not the **UNS**-property. This answers the question posed in [6] in the negative (see also Remark 6.1 (a) in [11]).

5 A Necessary Condition for the UNS-Property on $\mathcal{S}^n$

Given a linear transformation L on $\mathcal{S}^n$, it is difficult to verify whether L has the **UNS**-property or not. In this section, we formulate a simple necessary condition for the **UNS**-property in terms of the matrix induced by the transformation.

Consider the canonical Jordan frame $\{E_1, E_2, \ldots, E_n\}$ in $\mathcal{S}^n$, where E_i is the matrix with one in the (i,i) slot and zeros elsewhere. With respect to this, any element $X \in \mathcal{S}^n$ will have its Peirce decomposition given in the usual matrix form $X = [x_{ij}]$. By considering the elements of X that are on or below the diagonal, we can create a column vector

$$\mathrm{Vech}(X) := [\, x_{11}\ x_{21}\ \cdots\ x_{n1}\ x_{22}\ \cdots\ x_{n2}\ \cdots\ x_{nn}\,]^T.$$

The mapping $X \mapsto \mathrm{Vech}(X)$ creates a linear isomorphism between $\mathcal{S}^n$ and $\mathbb{R}^{\frac{n(n+1)}{2}}$. Because of this, any linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$ can be considered as a linear transformation on $\mathbb{R}^{\frac{n(n+1)}{2}}$. Thus, it can be represented (with respect to the standard basis) as a matrix $M \in \mathbb{R}^{\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}}$. We call M, the matrix *induced* by L.

Proposition 5.1 *Suppose $L : \mathcal{S}^n \to \mathcal{S}^n$ have the* **UNS***-property. Then its induced matrix is a* **P***-matrix.*

Proof. By our assumption, for the canonical frame in $\mathcal{S}^n$, we have

$$\|L(X) + \sum_{1 \le i \le j \le n} d_{ij} x_{ij}\| \ge \alpha \|X\| \quad \forall\, d_{ij} \ge 0,\ \forall\, X = [x_{ij}] \in \mathcal{S}^n,$$

which implies

$$\|M\,\mathrm{Vech}[x_{ij}] + \mathrm{Vech}[d_{ij}] * \mathrm{Vech}[x_{ij}]\| \ge \beta \|\mathrm{Vech}[x_{ij}]\|$$

for some $\beta > 0$, for all $d_{ij} \ge 0$, and for all matrices $X \in \mathcal{S}^n$. Since this inequality is like (3), by Lemma 4.1 in [5], M is a **P**-matrix. $\square$

Example 5.1 As an illustration of the above, consider the linear transformation $L : \mathcal{S}^2 \to \mathcal{S}^2$ given by

$$L \begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} ax + by & -bx \\ -bx & cy + dz \end{bmatrix}.$$

Then

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ -b & 0 & 0 \\ 0 & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Since $M_{22} = 0$, M is not a **P**-matrix, hence L does not have the **UNS**-property. When $a > 0$, $d > 0$, and $b \neq 0$, L has the **P**-property (see Example 5.2 in [11]). Thus, this example shows that *the* **P***-property need not imply the* **UNS***-property.*

Example 5.2 As a second illustration, consider $L \in \mathrm{Aut}(\mathcal{L}^3)$. Then (recalling that the underlying space of $\mathcal{L}^3$ is $\mathbb{R}^3$), $L = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}$, where $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an orthogonal matrix (see [11]). Corresponding to this, we define the linear transformation $\widehat{L} := G^{-1}LG : \mathcal{S}^2 \to \mathcal{S}^2$, where

$$G : \begin{bmatrix} x & y \\ y & z \end{bmatrix} \to \left[\frac{x+z}{2}, \frac{x-z}{2}, y\right]^T$$

maps $\mathcal{S}^2$ to $\mathcal{L}^3$. An easy calculation shows that

$$\widehat{L} : \begin{bmatrix} x & y \\ y & z \end{bmatrix} \to \begin{bmatrix} \frac{x+z}{2} + a\left(\frac{x-z}{2}\right) + by & c\left(\frac{x-z}{2}\right) + dy \\ c\left(\frac{x-z}{2}\right) + dy & \frac{x+z}{2} - \left[a\left(\frac{x-z}{2}\right) + by\right] \end{bmatrix}.$$

For this $\widehat{L}$, the induced matrix M is given by

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{x+z}{2} + a\left(\frac{x-z}{2}\right) + by \\ c\left(\frac{x-z}{2}\right) + dy \\ \frac{x+z}{2} - \left[a\left(\frac{x-z}{2}\right) + by\right] \end{bmatrix} = \begin{bmatrix} \frac{1+a}{2} & b & \frac{1-a}{2} \\ \frac{c}{2} & d & -\frac{c}{2} \\ \frac{1-a}{2} & -b & \frac{1+a}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Since D is an orthogonal matrix, we consider two cases:

Case 1: $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Thus, the induced matrix of $\widehat{L}$ is

$$M = \begin{bmatrix} \frac{1+\cos\theta}{2} & \sin\theta & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{2} & -\cos\theta & -\frac{\sin\theta}{2} \\ \frac{1-\cos\theta}{2} & -\sin\theta & \frac{1+\cos\theta}{2} \end{bmatrix}.$$

Now suppose that L has the **UNS**-property on $\mathcal{L}^3$. Then, because $\mathcal{S}^2$ and $\mathcal{L}^3$ are isomorphic, $\widehat{L}$ has the **UNS**-property on $\mathcal{S}^2$. By Proposition 5.1, the induced matrix of $\widehat{L}$ must be a **P**-matrix. Thus, M is a **P**-matrix, so $M_{22} = -\cos\theta > 0$, and

$$0 < \det \begin{bmatrix} \frac{1+\cos\theta}{2} & \frac{1-\cos\theta}{2} \\ \frac{1-\cos\theta}{2} & \frac{1+\cos\theta}{2} \end{bmatrix} = \cos\theta.$$

This contradiction indicates that Case 1 cannot happen.

Case 2: $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Hence, the induced matrix of $\widehat{L}$ is

$$M = \begin{bmatrix} \frac{1+\cos\theta}{2} & \sin\theta & \frac{1-\cos\theta}{2} \\ -\frac{\sin\theta}{2} & \cos\theta & \frac{\sin\theta}{2} \\ \frac{1-\cos\theta}{2} & -\sin\theta & \frac{1+\cos\theta}{2} \end{bmatrix}.$$

Suppose now that L has the **UNS**-property on $\mathcal{L}^3$. Then again, the induced matrix of $\widehat{L}$ must be a **P**-matrix. Thus, $M_{22} > 0$, that is, $\cos\theta > 0$. This implies that $D + D^T = \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix}$ is positive definite. Hence, L on $\mathbb{R}^3$ is positive definite (that is, strictly monotone). As strict monotonicity implies the **UNS**-property, we have arrived at the following:

An algebra automorphism of $\mathcal{L}^3$ has the **UNS***-property if and only if it is strictly monotone.*

It would be interesting to see if such a result holds in any $\mathcal{L}^n$.

6 Concluding Remarks

In this paper, we showed that the **UNS**-property is inherited by the principal subtransformations and, on simple algebras, it is invariant under the action of cone automorphisms. We also proved that on simple algebras, the **UNS**-property implies the **GUS**-property. These results allowed us to answer the question of Chua, Liu, and Yi [6] in the negative.

References

1. Fiedler, M., Pták, V.: On matrices with non-positive off-diagonal elements and positive principal minors. Czechoslovak Math. Jour. 12, 382–400 (1962)
2. Berman, A., Plemmons, R.J.: Nonnegative Matrices in the Mathematical Sciences. SIAM, Philadelphia (1994)
3. Cottle, R.W., Pang, J.-S., Stone, R.: The Linear Complementarity Problem. Academic Press, Boston (1992)
4. Gowda, M.S., Sznajder, R., J. Tao, J.: Some P-properties for linear transformations on Euclidean Jordan algebras. Linear Alg. Appl. 393, 203–232 (2004)

5. Chua, C.B., Yi, P.: A continuation method for nonlinear complementarity problems over symmetric cones. SIAM J. Optim. 20, 2560–2583 (2010)
6. Chua, C.B., Lin, H., Yi, P.: Uniform nonsingularity and complementarity problems over symmetric cones. Research Report, School of Physical & Mathematical Sciences, Nanyang Technological University, Singapore, (2009)
7. Faraut, J., Korányi, A.: Analysis on Symmetric Cones. Clarendon Press, Oxford (1994)
8. Gowda, M.S., Song, Y.: On semidefinite linear complementarity problems. Math. Prog. Series A 88, 575–587 (2000)
9. Tao, J.: Some P-properties of linear transformations on the Lorentz cone. Ph.D. Thesis, University of Maryland Baltimore County (2004)
10. Gowda, M.S., Song, Y., Ravindran, G.: On some interconnections between strict monotonicity, globally uniquely solvable, and P properties in semidefinite linear complementarity problems. Linear Alg. Appl. 370, 355–368 (2003)
11. Gowda, M.S., Sznajder, R.: Automorphism invariance of P and GUS properties of linear transformations on Euclidean Jordan algebras. Math. Oper. Res. 31, 109–123 (2006)
12. Gowda, M.S., Sznajder, R.: Some global uniqueness and solvability results for linear complementarity problems. SIAM J. Optim. 18, 461–481 (2007)
13. Zhou, Y., Gowda, M.S.: On the finiteness of the cone spectrum of certain linear transformations on Euclidean Jordan algebras. Linear Alg. Appl. 431, 772–782 (2009)