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WEAK UNIVALENCE AND CONNECTEDNESS OF INVERSE IMAGES OF CONTINUOUS FUNCTIONS

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A continuous function f with domain X and range $f(X)$ in R^n is weakly univalent if there is a sequence of continuous one-to-one functions on X converging to f uniformly on bounded subsets of X. In this article, we establish, under certain conditions, the connectedness of an inverse image $f^{-1}(q)$. The univalence results of Radulescu-Radulescu, Moré-Rheinboldt, and Gale-Nikaido follow from our main result. We also show that the solution set of a nonlinear complementarity problem corresponding to a continuous P_0-function is connected if it contains a nonempty bounded clopen set; in particular, the problem will have a unique solution if it has a locally unique solution.

1. Introduction. This article deals with the question of when, for a given continuous function f from a space into itself and for a vector q in the range of f, the inverse image $f^{-1}(q)$ is connected.

Our motivation for this study comes from linear complementarity problems. Given a matrix $M \in R^{n\times n}$ and a vector $q \in R^n$, the linear complementarity problem LCP (M, q) is to find a vector $x \in R^n$ such that

$$x \geq 0, \qquad Mx + q \geq 0, \qquad \langle Mx + q, x\rangle = 0. \tag{1}$$

This problem was proposed in the 1960s as a model for unifying linear/quadratic programming problems and the bimatrix game problem. LCP and its various generalizations (such as the nonlinear complementarity problem, vertical, horizontal, and mixed complementarity problems) have now found numerous applications in engineering, geometry, economics, and other areas, see for example, Cottle, Pang, and Stone (1992), Ferris and Pang (1997), Isac (1992) and Murty (1988). The variational inequality problem, which includes LCP as a special case, is equally important in various equilibrium settings and in applied mathematics.

In a recent article, Jones and Gowda (1998) showed that for a P_0-matrix M (that is, all principal minors of M are nonnegative), the solution set of LCP (M, q) is connected whenever it has a bounded connected component. As a consequence of this, they showed that the solution set is connected when it is bounded and that the problem has a unique solution when it has an isolated solution. These results were proved by formulating the LCP as a piecewise affine equation

$$F(x) := x \wedge (Mx + q) = 0, \tag{2}$$

where '$\wedge$' denotes the componentwise minimum of vectors involved. What happens if the affine function $Mx + q$ is replaced by a nonlinear continuous function ϕ satisfying the P_0-property? Can results of the above type be proved in other generalizations of the LCP? For the variational inequality problem? These motivated us to the question posed at the

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beginning of the Introduction. It is not clear to us whether such a question has been addressed in the finite dimensional optimization literature. But in the infinite dimensional (Banach space) setting (while describing the fixed point set of an operator) this question has been addressed by many researchers, see for example, Theorem 18.4, Exercises 18.6–18.9 in Deimling (1985), Theorem 48.2 and Remarks 48.6 in Krasnoselskii and Zabreiko (1984). Motivated by applications to integral/differential equations (for example, the Kneser-Fukuhara Theorem on page 315 of Krasnoselskii and Zabreiko 1984), these researchers assume that the function f can be approximated by functions which are "one-to-one" and deduce, under certain additional assumptions, that the inverse image is connected.

Since our focus here is finite dimensional problems, we restrict our attention to a function f with domain X and range $f(X)$ in R^n. We shall say that f is *weakly univalent* if it is *continuous* and there exists a sequence of *univalent* (i.e., one-to-one and continuous) functions f_k from X into R^n such that f_k converges to f uniformly on bounded subsets of X. Univalent functions, affine functions, monotone, and more generally P_0-functions on R^n are weakly univalent, see §4. The finite dimensional version of Theorem 48.2 in Krasnoselskii and Zabreiko (1984) for a weakly univalent function can be stated as follows:

THEOREM 1. *(Krasnoselskii-Zabreiko). Let Ω be a bounded open set in R^n, $f\colon \overline{\Omega} \to R^n$ be weakly univalent, and $q \in f(\overline{\Omega})\backslash f(\partial\Omega)$. Then $f^{-1}(q)$ is connected.*

In this paper, we generalize this result by dropping the boundedness hypothesis on Ω and by assuming the existence of a compact clopen subset of $f^{-1}(q)$ instead of the compactness of the entire $f^{-1}(q)$. As a consequence of this result, we deduce the well known Radulescu-Radulescu univalence theorem which includes the Moré-Rheinboldt, Gale-Nikaido univalence theorems, and the Hurwitz theorem in analytic function theory. As an application to optimization, we show that the nonlinear complementarity problem corresponding to a P_0-function has a globally unique solution if and only if it has a locally unique solution.

In the last section, we state an infinite dimensional analog of our main theorem.

2. Preliminaries. For properties of degree of a continuous function, we refer to Lloyd (1978). In R^n, the inner product of two vectors is denoted by $\langle x, y\rangle$. The interior, closure, and the boundary of a set D will be denoted by $\mathrm{int}(D)$, $\overline{D}$, and $\partial(D)$, respectively. The distance between a set D and a point x is defined by $d(x, D) := \inf\{\|x - y\| : y \in D\}$. A set E contained in another set F is said to be clopen in F if it is both open and closed in F.

3. The main result.

THEOREM 2. *Let X be a subset of R^n with nonempty interior, $f : X \to R^n$ be weakly univalent, and $q \in f(X)$. Suppose that there is a nonempty set E such that*

(a) E is compact and clopen in $f^{-1}(q)$, and

(b) $E \subset \mathrm{int}(X)$.

Then $f^{-1}(q) = E$ and is connected.

PROOF. We first show that $f^{-1}(q) = E$. Assume the contrary and let, without loss of generality, $q = 0$, and write

$$S := f^{-1}(0) \quad \text{and} \quad \Omega := \mathrm{int}(X).$$

We first establish the existence of a bounded open set D in Ω such that

$$E \subset D \subset \overline{D} \subset \Omega \quad \text{and} \quad \overline{D} \cap (S\backslash E) = \varnothing. \tag{3}$$

In order to see this, we let $F := \Omega^c \cup \overline{S\backslash E}$, where Ω^c denotes the complement of Ω in R^n.

Then E and F are disjoint. (Indeed, since $E \subset \Omega$, E is disjoint from Ω^c. If $x \in \overline{S \backslash E} \cap \Omega$, then $x \in S\backslash E$ by the openness of E in S, and so cannot be in E.) Now E and F are disjoint closed subsets of R^n and hence there exists an open set D such that $E \subset D \subset \overline{D} \subset F^c$; see e.g., page 144, Dugundji (1970). This gives (3). Since E is compact, we can assume that D is bounded. We note that $\partial D \cap S = \varnothing$, i.e.,

(4) $$0 \notin f(\partial D) \quad \text{and hence} \quad d(0, f(\partial D)) > 0.$$

Let $x^* \in E$ and $\bar{x} \in S\backslash E$. Let f_k be a sequence of univalent functions on X converging to f uniformly on bounded subsets of X. Let

$$G_k(x) = f_k(x) - f_k(x^*) \quad \text{and} \quad H_k(x) = f_k(x) - f_k(\bar{x}).$$

Since $f_k(x^*) \to f(x^*) = 0$ and $f_k(\bar{x}) \to f(\bar{x}) = 0$, by the uniform convergence of G_k and H_k on $\overline{D}$, we can take a large k so that

(5) $$\sup_{\overline{D}} \|G_k(x) - f(x)\| < d(0, f(\partial D)) \quad \text{and} \quad \sup_{\overline{D}} \|H_k(x) - f(x)\| < d(0, f(\partial D)).$$

It follows from the nearness property of the degree (Theorem 2.1.2, Lloyd 1978) that

(6) $$\deg(H_k, D, 0) = \deg(f, D, 0) = \deg(G_k, D, 0).$$

But $\deg(G_k, D, 0) = \pm 1$, since G_k is one-to-one on D and $G_k(x^*) = 0$ (Theorem 3.3.3, Lloyd 1978). Hence $\deg(H_k, D, 0) \neq 0$, implying the existence of a zero of H_k in D (Theorem 2.1.1, Lloyd 1978). Since the unique zero of H_k is $\bar{x}$ which is outside of D, we have a contradiction. Thus we have proved that

$$S = E.$$

To prove the connectedness of S, suppose there is a clopen subset W in S. Since now S $(=E)$ is compact, W is compact and contained in Ω. Repeating the above argument for W in place of E, we see that $S = W$. Thus S is connected. This completes the proof. □

The following example shows that the compactness of E is essential in the theorem.

EXAMPLE. Let $X = \Omega = \{(x, y) \in R^2 : |x| < 1,\ 0 < |y| < 1\}$, $f(x, y) = (x, 0) : \Omega \to R^2$, and $f_k(x, y) = (x, y/k)$. Then $f_k \to f$ uniformly on bounded subsets of Ω and each f_k is univalent. The set $E = \{(0, y) \in \Omega : y > 0\}$ is clopen in $f^{-1}(0, 0)$ but not compact. The set $f^{-1}(0, 0)$ is not connected in Ω.

REMARKS. (1) For $X = R^n$, conditions (a) and (b) of the theorem are satisfied when $E = f^{-1}(q)$ is compact. We conclude that for a weakly univalent function $f : R^n \to R^n$, $f^{-1}(q)$ is connected whenever it is bounded.

(2) As a by-product of the above proof, we get the following: if $f : R^n \to R^n$ is weakly univalent, and E is a nonempty bounded clopen subset of $f^{-1}(q)$, then there exists a bounded open set D containing E such that

$$\deg(f, D, q) = \pm 1.$$

In particular, this holds when $f^{-1}(q)$ is nonempty and compact. Such a property is useful in studying the stability aspects of the equation $f(x) = q$.

(3) When $f : R^n \to R^n$ is a piecewise affine function (that is, f is a continuous selection

of a finite number of affine functions), every inverse image is a finite union of polyhedral sets. In this setting, an inverse image will have a nonempty bounded clopen set if and only if it has a compact connected component. This is especially true for the function defined in (2).

4. Some uniqueness and univalence theorems.

THEOREM 3. *Let Ω be an open set in R^n, $f: \Omega \to R^n$ be weakly univalent, and $q \in f(\Omega)$. Then $f^{-1}(q)$ is a singleton under any one of the following conditions:*

(a) $f^{-1}(q)$ is zero dimensional, i.e., any open set in $f^{-1}(q)$ is a union of clopen sets in $f^{-1}(q)$.

(b) $f^{-1}(q)$ is countable.

(c) $f^{-1}(q)$ has an isolated point.

PROOF. When (a) holds, we can produce a nonempty compact clopen subset of $f^{-1}(q)$ by intersecting $f^{-1}(q)$ with an appropriate open ball in R^n. (For example, let $a \in f^{-1}(q)$, $r > 0$ such that the closure of the open ball $B(a, r)$ is contained in Ω. Let V be a clopen subset of $f^{-1}(q)$ contained in $f^{-1}(q) \cap B(a, r)$. It is easily seen that V is closed in R^n. Being bounded, it is compact.) By Theorem 2, $f^{-1}(q)$ is connected. Since it is zero dimensional, it must be a singleton set (page 15, Hurewicz and Wallman 1948).

Condition (b) is a special case of (a) (page 10, Hurewicz and Wallman 1948). Finally, when (c) holds, $f^{-1}(q)$ will have a singleton set as a compact clopen subset, and Theorem 2 is applicable. □

The following results are immediate.

THEOREM 4. (RADULESCU AND RADULESCU 1980). *Suppose that Ω is an open set in R^n, f: $\Omega \to R^n$ is weakly univalent and light (i.e., for each $q \in f(\Omega)$, $f^{-1}(q)$ is zero dimensional). Then f is univalent.*

We remark that the actual Radulescu-Radulescu theorem is stated in a Banach space setting.

THEOREM 5. (MORÉ AND RHEINBOLDT 1973). *Suppose that Ω is an open set in R^n, $f: \Omega \to R^n$ is continuous and for each $\epsilon > 0$, $f_\epsilon(x) := f(x) + \epsilon x$ is univalent on Ω. If f is differentiable on Ω and the Jacobian matrix $f'(x)$ is nonsingular at each point of Ω, or more generally, if each point $x^* \in \Omega$ has a neighborhood in which the equation $f(x) = f(x^*)$ has only one solution, then f is univalent.*

For an excellent introduction to the extensive literature on univalent theorems, see Parthasarathy (1983).

We say that a function $\phi : R^n \to R^n$ is a *P_0-function* if for every pair x and y with $x \neq y$ in R^n, we have

$$\max_{\{i:x_i \neq y_i\}} (\phi(x) - \phi(y))_i(x - y)_i \geq 0.$$

By replacing the inequality '≥' by '>,' we get the definition of a P-function. When $\phi(x) = Mx + q$ with $M \in R^{n\times n}$, these definitions reduce to those of a P_0-matrix and a P-matrix. Also, when ϕ is differentiable, ϕ is a P_0 (P)-function if and only if the Jacobian matrix $\phi'(x)$ is a P_0 (respectively, a P)-matrix at each x, see Moré and Rheinboldt (1973). It is obvious that if ϕ is a P_0-function, then it is weakly univalent since $\phi(x) + \epsilon x$ is a P-function for each $\epsilon > 0$, and hence univalent.

We mention a famous consequence of Theorem 5 stated for R^n (although the result is valid for any open rectangle in R^n), see Moré and Rheinboldt (1973).

THEOREM 6. (GALE-NIKAIDO 1965). *Suppose that f is differentiable on R^n and the Jacobian matrix $f'(x)$ is a nonsingular P_0-matrix for each x. Then f is univalent.*

We end this section by noting that the Hurwitz theorem of analytic function theory (that if a sequence of univalent analytic functions on an open connected set in the complex plane converges to a function uniformly on compact subsets, then the limit function is either univalent or a constant) is a consequence of Theorem 4.

5. Applications to complementarity problems. Consider the nonlinear complementarity problem NCP(ϕ, q) corresponding to a function $\phi : R^n \to R^n$ and a vector $q \in R^n$: find x such that

$$x \geq 0, \qquad \phi(x) + q \geq 0, \qquad \langle \phi(x) + q, x \rangle = 0, \tag{7}$$

or equivalently,

$$x \wedge [\phi(x) + q] = 0.$$

When ϕ is monotone, i.e.,

$$\langle \phi(x) - \phi(y), x - y \rangle \geq 0, \qquad \forall\, x, y \in R^n,$$

it can be easily shown that the solution set NCP(ϕ, q) is closed and convex. (The generalized equation formulation of NCP(ϕ, q) leads to a maximal monotone operator (cf. Robinson 1979) and any such operator will have closed convex preimages, see, Aubin and Frankowska 1990, Proposition 3.5.6.) Since every monotone function is a P_0-function, it is natural to ask whether the solution set of NCP(ϕ, q) corresponding to a P_0-function is at least connected. Unfortunately, this need not be true even when the function is linear, as demonstrated by an example due to Stone, see Jones and Gowda (1998). However, as we see below, Theorem 2 can be used to derive a connectedness result extending a similar result for P_0-matrices.

THEOREM 7. *Suppose ϕ is a continuous P_0-function and $q \in R^n$. If the solution set of NCP(ϕ, q) contains a nonempty bounded clopen subset, then the solution set is connected. In particular, the solution set is connected if it is bounded, and NCP(ϕ, q) has a unique solution if it has an isolated solution.*

PROOF. Without loss of generality, let $q = 0$. We put $f(x) = x \wedge \phi(x)$ and observe that NCP(ϕ, 0) is equivalent to solving the equation $f(x) = 0$. We show that f is weakly univalent and apply Theorem 2. For each natural number k, the function $\phi(x) + x/k$ is a P-function and $f_k(x) := x \wedge [\phi(x) + x/k]$ converges to f uniformly on bounded subsets of R^n as $k \to \infty$. Fix k, let $\psi(x) = \phi(x) + x/k$ and $g(x) := x \wedge \psi(x)$. For any r, the equation

$$x \wedge \psi(x) = r \tag{8}$$

can be rewritten, via the transformation $x - r = u$, as

$$u \wedge [\psi_r(u) - r] = 0 \tag{9}$$

where $\psi_r(u) = \psi(u + r)$. Since $u \mapsto \psi_r(u) - r$ is a P-function, it is elementary and well known (Theorem 2.3, Moré 1974) that (9) has at most one solution u; hence (8) has at most one solution x. Thus g is univalent and hence f is weakly univalent. An application of Theorem 2 completes the proof. □

REMARKS. The above theorem (and its proof) can be modified to get a similar result for vertical nonlinear complementarity problems and for mixed nonlinear complementarity problems.

(1) Given functions $\phi_1, \phi_2, \ldots, \phi_k$ from R^n into itself, the *vertical nonlinear complementarity problem* (VNCP), denoted by VNCP $(\phi_1, \phi_2, \ldots, \phi_k)$, is to find an $x \in R^n$ such that

$$x \wedge \phi_1(x) \wedge \cdots \wedge \phi_k(x) = 0. \tag{10}$$

Let $\phi : R^n \to R^n$ be a *row representative* of $\{\phi_1, \phi_2, \ldots, \phi_k\}$, which means that for each x and $i = 1, 2, \ldots, n$, $(\phi(x))_i \in \{(\phi_1(x))_i, \ldots, (\phi_k(x))_i\}$. Let us say that $\{\phi_1, \phi_2, \ldots, \phi_k\}$ has the P_0 (P)-property if every row representative is a P_0-function (respectively, a P-function). We can state the previous theorem (with a similar proof) for a VNCP whose corresponding set of functions has the P_0-property. We omit the details except demonstrate the fact that (10) has at most one solution when $\{\phi_1, \phi_2, \ldots, \phi_k\}$ has the P-property. So assume this P-property and suppose we have $x \neq y$ such that

$$x \wedge \phi_1(x) \wedge \phi_2(x) \wedge \cdots \wedge \phi_k(x) = 0 = y \wedge \phi_1(y) \wedge \phi_2(y) \wedge \cdots \wedge \phi_k(y).$$

For each index i, we pick ϕ_{i^*} as follows: if $(x - y)_i > 0$, then $x_i > 0$ so that there exists $(\phi_{i^*}(x))_i = 0$ in which case, $(\phi_{i^*}(x) - \phi_{i^*}(y))_i \leq 0$. If $(x - y)_i < 0$, we proceed as before by reversing the roles of x and y. This results in $(\phi_{i^*}(x) - \phi_{i^*}(y))_i \geq 0$. If $(x - y)_i = 0$, then pick any ϕ_{i^*}. Denoting the resulting row representative by ϕ, we see that

$$\max_{\{i : x_i \neq y_i\}} (x - y)_i (\phi(x) - \phi(y))_i \leq 0$$

which contradicts the P-property of the representative ϕ. Thus when the set of functions defining the VNCP has the P-property, the VNCP has at most one solution.

(2) Now consider the *mixed nonlinear complementarity problem* (MNCP) defined by the equation

$$\begin{bmatrix} f(x, y) \\ y \wedge g(x, y) \end{bmatrix} = 0 \tag{11}$$

where $f : R^m \times R^n \to R^m$ and $g : R^m \times R^n \to R^n$ are given functions. Under the assumption that $(x, y) \mapsto (f, g)$ is a continuous P_0-function in (x, y), we can easily show (as in the proof of Theorem 7) that the function defining (11) is weakly univalent. A result similar to Theorem 7 can now be stated for the MNCP. We omit the details.

6. An infinite dimensional analog of the main theorem. Let B be a Banach space, X be a set in B with nonempty interior. Recall that a continuous function $h : X \to B$ is *compact* (or completely continuous) if for each bounded set A in X, $\overline{h(A)}$ is compact in B. Let $\mathcal{M}(X)$ denote the set of all compact perturbations of the identity map on B, i.e., $f \in \mathcal{M}(X)$ if there exists a compact function h on X such that $f(x) = x - h(x)$ for all $x \in X$. Note that when $B = R^n$, every continuous function on a closed set X is in $\mathcal{M}(X)$.

THEOREM 8. *Let X be a subset of B with* $\text{int}(X) \neq \emptyset$, $f \in \mathcal{M}(X)$, *and* $q \in f(X)$. *Suppose further that there exists a sequence (f_k) in $\mathcal{M}(X)$ such that each f_k is one-to-one and $f_k \to f$ uniformly on bounded subsets of X. If $f^{-1}(q)$ has a nonempty subset E such that*

(a) *E is compact and clopen in $f^{-1}(q)$, and*

(b) $E \subset \text{int}(X)$,

then $f^{-1}(q) = E$ and is connected.

The proof is similar to that of Theorem 2. We see (4) even in this infinite dimensional setting. That $d(0, f(\partial D)) > 0$ follows from $0 \notin f(\partial D)$ and the compactness of h in $f(x) = x - h(x)$. To go from (5) to (6), we quote Theorem 4.3.7 in Lloyd (1978) instead of Theorem 2.1.2. The equality $\deg(G_k, D, 0) = \pm 1$ follows from Theorem 4.3.14, Lloyd (1978).

7. Concluding remarks. In this paper, we have presented a result on the connectedness of an inverse image of a weakly univalent function defined on a subset of R^n. In addition to deducing various uniqueness and univalence results from this (main) result, we showed that for the nonlinear complementarity problem corresponding to a continuous P_0-function, the solution set is connected if it contains a bounded clopen subset and that the solution set is singleton if it contains an isolated solution. We note that similar NCP results were obtained recently for a continuously differentiable P_0-function by Facchinei (1997) via the Mountain Pass Theorem.

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