The Fundamental Theorem of Calculus

3.1 The Shoulders of Giants

Among the many superb contributions to mathematics made by Sir Isaac Newton (1642-1727), probably the most far-reaching was his invention of a new system of analysis called the "method of fluxions". This new method of reckoning with functions made it possible to solve all sorts of hitherto intractable problems pertaining to continuous rates of change. Newton's method of fluxions quickly evolved into the branch of mathematics known as the Calculus.

A famous quotation, attributed to Newton, asserts:

If I have managed to see farther than others,
it was only by standing on the shoulders of giants.

One of those giants was Sir Isaac Barrow (1630-1677), regarded by many as Newton's mentor. Barrow was a multi-gifted individual who achieved recognition in numerous fields, including mathematics, physics, astronomy, theology and the Greek classics. In his book *Lectiones Opticae et Geometricae* (1669), Barrow states and proves a theorem which, in essence, equates the operation of integration to the operation of antidifferentiation. In view of its importance, this theorem has come to be known as the Fundamental Theorem of Calculus.

3.2 A Brief Review of Derivatives

Let f denote a real-valued function of a real variable. The *derivative* of f, denoted by f' or by $\frac{d}{dx}(f)$, is the function given by the formula

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.}$$

For a particular value of x, the above limit may or may not exist. If it exists, then the function f is said to be *differentiable* at x. Otherwise, the derivative function f' is considered to be undefined at that particular value of x.

The quotient in the above limit is refered to by some as *Newton's quotient*. Graphically, Newton's quotient can be interpreted as the slope of the *secant line* passing through the points $P_0 = (x, f(x))$ and $P_h = (x+h, f(x+h))$ on the graph of $y = f(x)$. As $h \to 0$, the point P_h approaches the point P_0. Correspondingly, the secant line through the points P_h and P_0 merges with the *tangent line* to the graph at the point P_0. Consequently, the slope m of the tangent line to the graph of $y = f(x)$ at the point $(x, f(x))$ is given by the formula

$$m = f'(x)$$

Ex: Using only the basic definitions, let us find the equation of the tangent line to the graph of $y = f(x) = x^2 - 3x$ at the point $(5, 10)$. Using the definition of $f'(x)$ as a Newton quotient, we have:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h} \\ &= \lim_{h \to 0} \frac{2hx - 3h + h^2}{h} \\ &= \lim_{h \to 0} (2x - 3 + h) \\ &= 2x - 3 \end{aligned}$$

Thus, if the tangent line at the point $(5, 10)$ is assumed to have the equation $y = mx + b$, then $m = f'(5) = 7$. So the equation of the tangent line is $y = 7x + b$. To find b, we use the fact that the tangent line passes through the point $(5, 10)$. Hence $10 = 7(5) + b$, from which it follows that $b = -25$. Therefore, the equation of the tangent line is $y = 7x - 25$.

Viewed as the slope of the tangent line to the graph of $y = f(x)$, the derivative $f'(x)$ captures a lot of important information about the behavior of the function $f(x)$. For instance, if $f'(x) > 0$ then f must be increasing at x. Similarly, if $f'(x) < 0$ then f must be decreasing at x. On the other hand, if $f'(x) = 0$, then the tangent line to the graph of $y = f(x)$ must be horizontal at x. Conversely, in order for the function f to possess a local maximum or a local minimum at a given point, the tangent line to the graph of $y = f(x)$ at that point must be either horizontal ($f'(x) = 0$) or undefined ($f'(x) = \nexists$). The latter observation can be usefully applied to solving optimization problems.

Ex: As in the previous example, consider the function $f(x) = x^2 - 3x$, for which we showed that $f'(x) = 2x - 3$. To determine the x-values over which f is increasing, we must solve the inequality $f'(x) > 0$, or $2x - 3 > 0$. The solution set, $x > \frac{3}{2}$, means that f is increasing over the x-interval $(\frac{3}{2}, \infty)$. Similarly, solving the inequality $f'(x) < 0$, or $2x - 3 < 0$, we conclude that $f(x)$ is decreasing over the x-interval $(-\infty, \frac{3}{2})$. To find the local extrema (maxima or minima) of f, we need only examine the x-values where $f'(x) = 0$ or undefined. The only solution is $x = \frac{3}{2}$. Since f is decreasing from the left and increasing to the right of $x = \frac{3}{2}$, we conclude that $x = \frac{3}{2}$ corresponds to a local minimum of f. In fact, in this case, it gives the absolute minimum $f(\frac{3}{2}) = -\frac{9}{4}$.

For applications in science, engineering, economics, and other areas of practical interest, it is natural to think of the derivative as a measure of the *instantaneous rate of change* of one variable quantity relative to another. Suppose two variables y and t are related by the formula $y = f(t)$. As t varies from an initial value $t = a$ to a terminal value $t = b$, the value of y varies correspondingly from $y = f(a)$ to $y = f(b)$. By definition, the *average rate of change* of y relative to t over the t-interval $[a, b]$ is given by the quotient

$$\frac{\Delta y}{\Delta t} = \frac{f(b) - f(a)}{b - a} = \frac{f(a+h) - f(a)}{h}$$

where $\Delta t = b - a = h$ and $\Delta y = f(b) - f(a) = f(a+h) - f(a)$. The instantaneous rate of change of y relative to t at the instant when $t = a$ is obtained by taking the limit of the above expression as $h \to 0$, yielding the derivative $f'(a)$.

Ex: In studies involving motion, velocity is defined as the instantaneous rate of change of distance relative to time. For instance, imagine we have programmed a vehicle to move along a straight track in such a manner that, at any time t, its distance s from a fixed initial point on the track is given by the formula $s(t) = t^2 - 3t$. By the above example, we have $s'(t) = 2t - 3$. Therefore the velocity $v(t)$ of the vehicle at any time t is $v(t) = 2t - 3$.

Computing derivatives by the Newton quotient can be quite cumbersome, especially for functions more complicated than just ordinary polynomials of very low degree. Fortunately, thanks to the well-known propensity of mathematicians to cater to lazy instincts, general formulas have been derived that enable us to circumvent, once and for all, the need to differentiate specific functions by means of the Newton quotient. For instance, using the Newton quotient, mathematicians have shown that for any function of the form $f(x) = x^s$, $s \neq 0$, its derivative is given by $f'(x) = sx^{s-1}$. Thus, never again do we have to bother with the Newton quotient when treating specific functions of this form. Similarly, using the Newton quotient, general formulas have been derived for the derivatives of various combinations of functions ($f + g$, fg, $f \circ g$, etc.) in terms of the derivatives of the component functions. However, as self-respecting students of mathematics, let us not lose sight of the fact that, in order to derive these formulas (collected below), someone had to deal directly with the Newton quotient.

GENERAL DERIVATIVE FORMULAS

Let f and g be functions, and let α and β be real numbers.

[GDF 1] $(\alpha f + \beta g)' = \alpha f' + \beta g'$ *(Linearity Rule).*

[GDF 2] $(fg)' = f'g + fg'$ *(Product Rule).*

[GDF 3] $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$ *(Quotient Rule).*

[GDF 4] $(f \circ g)' = (f' \circ g)g'$ *(Chain Rule).*

[GDF 5] $\left(f^{-1}\right)' = \dfrac{1}{f' \circ f^{-1}}$ *(Inverse Function Rule).*

DERIVATIVES OF SPECIAL FUNCTIONS

$f(x)$	$f'(x)$
k (const.)	0
x^s ($s \neq 0$)	sx^{s-1}
e^x	e^x
$\ln(x)$	$1/x$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

Using the above formulas, we can differentiate just about any function that a first course in calculus can visit upon us. In fact, using the above formulas, we can derive our own formulas for the derivatives of some important special functions not appearing in the above table.

Ex: The following additional formulas are easily derived:

(a) $\frac{d}{dx}(\tan(x)) = \sec^2(x)$

(b) $\frac{d}{dx}(a^x) = \ln(a)a^x$

(c) $\frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a)x}$

(d) $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$

To derive formula (a), recall that $\tan(x) = \sin(x)/\cos(x)$. By the quotient rule, we get

$$\frac{d}{dx}\tan(x) = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

To derive formula (b), note that $a^x = e^{\ln(a)x}$. By the chain rule, we get

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{\ln(a)x}\right) = \ln(a)e^{\ln(a)x} = \ln(a)a^x\,.$$

To derive formula (c), recall that $\log_a(x) = \ln(x)/\ln(a)$. Thus

$$\frac{d}{dx}(\log_a(x)) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right) = \frac{1}{\ln(a)}\frac{d}{dx}(\ln(x)) = \frac{1}{\ln(a)}\frac{1}{x} = \frac{1}{\ln(a)x}\,.$$

To derive formula (d), apply the inverse function rule with $f(x) = \sin(x)$. We get

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\cos\left(\sin^{-1}(x)\right)} = \frac{1}{\sqrt{1-x^2}}\,.$$

3.3 A Brief Review of Antiderivatives

Literally, antidifferentiation is differentiation in reverse. More precisely, if two functions f and F are related by the formula $\frac{d}{dx}F(x) = f(x)$, then F is called an *antiderivative* of f.

Ex: Let $f(x) = 3x^2 + 2$. One possible antiderivative of f is the function $F(x) = x^3 + 2x$. Another possible antiderivative is the function $F(x) = x^3 + 2x + 10$. In fact any function of the form $F(x) = x^3 + 2x + C$, where C is a constant, is an antiderivative of f.

As illustrated by the above example, a given function f admits many possible antiderivatives F, any two of which differ by an additive constant. To represent the phenomenon of multiple antiderivatives, we introduce the idea of the *most general antiderivative* of f, also called the *indefinite integral* of f, symbolized by an expression of the form $\int f(x)\,dx$. Accordingly, if F is any particular antiderivative of f (i.e. $F'(x) = f(x)$), then we can write

$$\int f(x)\,dx = F(x) + C$$

where the symbol C denotes an undetermined constant (independent of x).

Since antidifferentiation is defined as the inverse operation to differentiation, every rule or formula pertaining to derivatives yields a corresponding rule or formula pertaining to antiderivatives. Below we collect some of the more important rules and formulas pertaining to antiderivatives.

GENERAL ANTIDERIVATIVE FORMULAS

Let f and g be functions, and let α and β be real numbers.

[GADF 1] $\displaystyle\int (\alpha f(x) + \beta g(x))\, dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx$ *(Linearity Rule).*

[GADF 2] $\displaystyle\int f(x) g'(x)\, dx = f(x)g(x) - \int g(x) f'(x)\, dx$ *(Integration by Parts).*

[GADF 3] $\displaystyle\int f(g(x))g'(x)\, dx = \int f(u)\, du$ where $u = g(x)$ *(u-substitution Rule).*

ANTIDERIVATIVES OF SPECIAL FUNCTIONS

$f(x)$	$\int f(x)\, dx$
k (const.)	$kx + C$
x^s $(s \neq -1)$	$\frac{1}{s+1} x^{s+1} + C$
$1/x$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\tan(x)$	$-\ln\lvert\cos(x)\rvert + C$
$1/(a^2 + x^2)$	$\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

In addition to the above rules and formulas, a number of so-called "integration techniques" can be used to find formulas for the antiderivatives of special kinds of functions. Probably the most palatable of these is a method called "integration by partial fractions" which pertains only to rational functions. A rational function is a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials. The method of partial fractions exploits the Fundamental Theorem of Algebra to express a given rational function as a sum of simple parts, each of which possesses an elementary antiderivative. It turns out that every rational function possesses an antiderivative which can be expressed as sum of rational, logarithmic and inverse-tan functions. For complete details on this and other integration techniques, the reader should consult a good calculus textbook such as Stewart's *Calculus–Early Transcendentals, 4-th Edition.*

As we saw in the previous section (3.2), every elementary function (of the kind appearing in a typical first course in calculus) possesses a derivative which is itself an elementary function. In general, the derivative of any function is a function of a type no more complicated than the function itself. The opposite is true of antiderivatives. All too often, the antiderivative of a given elementary function is not elementary at all, nor expressible as any simple algebraic combination of elementary functions. For instance, the antiderivative of the elementary function $f(x) = \sqrt{x^4 + 1}$ is in no way capable of being expressed in terms of elementary functions (of the kind encountered in a first course in calculus). In fact, only an exceedingly small minority (zero relative density) of elementary functions possess antiderivatives capable of being expressed by simple formulas constructed from elementary functions.

In your typical Calculus II course, the student is expected to expend considerable effort in wrestling with exercises of the form "find an explicit formula for the antiderivative $\int f(x)\,dx$". Fortunately for the student, all such exercises are carefully engineered in advance to succumb to the recipes (techniques of integration) covered in the course. Basically, if you just learn to recognize certain patterns and massage them in the appropriate way, then every exercise gives way to a happy ending. In real life, the situation is starkly different. Rarely do we encounter a function in science, engineering, economics or any other field of practical interest, whose antiderivative is capable of being expressed by a simple formula constructed from elementary functions. This might be why tables of integrals are so popular with scientists and engineers. For instance, in the famous book by Gradshteyn and Ryzhik entitled *Table of Integrals, Series and Products* the reader can feast on what is perhaps the world's most monumental collection of antiderivative formulas, literally numbering in the thousands, more than enough to satisfy even the most exuberant appetite. More recently, with the advent of efficient computational technologies, such tables have been incorporated into powerful software packages such as MATHEMATICA and MAPLE. Students planning a career in engineering or any of the natural sciences are well advised to acquaint themselves with these important tools.

3.4 A Brief Review of Integration

The definition of integration has nothing to do with derivatives or antiderivatives. The basic problem, which integration addresses, is how to determine the area of a region R in the xy-plane bounded above and below by the graphs of two functions $f(x)$ and $g(x)$. For simplicity, we can start by assuming that $g(x) = 0$, $f(x) \geq 0$, and that the region R, whose area we wish to compute, is confined above the x-interval $[a, b]$. That is, $R = \{(x, y) \mid a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$, which can be aptly illustrated by means of a picture. By definition,

$$\text{Area}(R) = \int_a^b f(x)\,dx.$$

It is important to understand that the above formula does not purport to represent anything other than the definition of the symbol $\int_a^b f(x)\,dx$, which, at first glance, might resemble an antiderivative. However, for the moment we cannot assume any connection whatever to derivatives or antiderivatives. This connection will emerge later.

Ex: Let $f(x) = k$, where $k > 0$. Then $\int_a^b k\,dx$ is equal to the area of a rectangle of height k and base $b - a$. Therefore $\int_a^b k\,dx = (b - a)k$. $\blacksquare$

Ex: To evaluate the integral $\int_0^5 \sqrt{25-x^2}\,dx$, we observe that the region R whose area is represented by this integral is the portion of the disk of radius 5 centered at the origin confined to the first quadrant of the xy-plane. Therefore, using the formula for the area of a circle, we get $\int_0^5 \sqrt{25-x^2}\,dx = \frac{1}{4}\pi(5^2) = \frac{25\pi}{4}$.

If the function f is reasonably well-behaved over the interval $[a,b]$ (i.e. does not possess too many discontinuities), then the area $\int_a^b f(x)\,dx$ of the region R can be approximated to any desired level of accuracy by means of a *Riemann sum.* We briefly describe the process below.

To begin, we subdivide the region R into a succession of thin vertical strips R_i of constant width Δx. Therefore the number of strips is $n = \frac{b-a}{\Delta x}$, and

$$\int_a^b f(x)\,dx = \text{Area}(R) = \sum_{i=1}^{n} \text{Area}(R_i).$$

At the base of each strip R_i, we choose an x-value x_i. If the function is reasonably continuous, then each strip R_i is closely matched in shape and size by a rectangle of height $f(x_i)$, width Δx and area $f(x_i)\Delta x$. Therefore $\text{Area}(R_i) \approx f(x_i)\Delta x$, from which it follows that

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\Delta x.$$

As the number n of strips increases and the width Δx of each of the strip decreases, the approximation becomes more and more accurate. In the final analysis, we get

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x.$$

There are a number of ways to carry out the above process, depending on the shape of the strips and the choice of the base points x_i. From a computational point of view, the least efficient method is to use rectangular strips with flat tops and to choose the base points x_i as either the leftmost or rightmost points of the base intervals. A slightly more efficient method uses rectangular strips with base points x_i located at the centers of the base intervals. More efficient still is to use trapezoidal strips with central base points. But for the purpose of high power computation, nothing less than *Simpson's Rule* is acceptable. Simpson's Rule uses strips whose curved tops are pieces of graphs of quadratic functions. At the top of each strip, the curved ceiling is designed to closely approximate the shape of the graph of $y = f(x)$.

To some students of mathematics it may come as surprise to learn that most integrals cannot be computed except by an approximation process like that described above. After learning about the connection between integration and antidifferentiation, many students conveniently forget the distinction between them, and treat every integration problem as if it were a problem in antidifferentiation. This is a totally unrealistic view of life. In practice, only a very tiny minority (zero relative density) of definite integrals can be computed exactly by way of antidifferentiation. For most functions, the luxury of being able to find a reasonably tame antiderivative is simply not an option.

3.5 The Fundamental Theorem of Calculus

The integral symbol $\int$ used to denote the most general antiderivative is identical symbol used to denote integration. Of course, this is no accident. However, unless we can find positive proof of a connection, there is no reason to presume that any connection exists between the idea of antidifferentiation and the idea of integration.

The connection comes about through the concept of the *area function*. For a given function f, which, for simplicity, we assume to be positive and continuous over the x-interval $[c, d]$, the area function $\mathcal{A}_f(x)$ is defined, for $c \le x \le d$, as the area of the region R_x bounded above by the graph of $y = f(x)$ and below by the x-interval $[c, x]$. The idea can be aptly illustrated with a picture. Thus, by definition

$$\mathcal{A}_f(x) = \text{Area}(R_x) = \int_c^x f(t)\,dt.$$

In the above integral, in order to avoid using the symbol x in two different roles, we have inserted the "dummy variable" t in place of the more customary dummy variable x. Note that x is being used as the argument of the area function.

Ex: To illustrate the definition of the area function, let us find an explicit formula for the area function of $f(x) = 2x$ over the interval $[0, x]$. For any $x > 0$, the region R_x is a triangle with base length x and height $2x$. Therefore the area of R_x is $\frac{1}{2}(x)(2x) = x^2$. Thus, $\mathcal{A}_f(x) = x^2$.

The Fundamental Theorem of Calculus is closely related to the following theorem.

Area Function Theorem: $\frac{d}{dx}\mathcal{A}_f(x) = f(x)$.

Proof: Let $h > 0$. We observe that the vertical strip S below the graph of $y = f(x)$ and above the interval $[x, x+h]$ is the difference between the regions R_x and R_{x+h}. Thus

$$\text{Area}(S) = \text{Area}(R_{x+h}) - \text{Area}(R_x) = \mathcal{A}_f(x+h) - \mathcal{A}_f(x)$$

On the other hand, since the graph is continuous, if h is small enough, then the area of the vertical strip S is very nearly equal to the area of the rectangle of height $f(x)$ sitting on the interval $[x, x+h]$. That is, $\text{Area}(S) \approx hf(x)$. It can be shown that this approximation becomes more and more accurate as $h \to 0$, in the sense that

$$\lim_{h \to 0} \frac{\text{Area}(S)}{h} = f(x).$$

Therefore

$$\lim_{h \to 0} \frac{\mathcal{A}_f(x+h) - \mathcal{A}_f(x)}{h} = f(x)$$

which, by the definition of the derivative, is equivalent to the assertion of the theorem. A similar argument can be crafted for $h < 0$. ■

The reader should take careful note of what the Area Function Theorem says. It says that the area function $\mathcal{A}_f(x)$ is an antiderivative for the function $f(x)$.

Now imagine we want to evaluate an integral of the form $I = \int_a^b f(x)\,dx$ where $c \le a < b \le d$. In this situation, the integral represents the area of a certain region which is the difference between the regions R_b and R_a. Therefore

$$\int_a^b f(x)\,dx = \text{Area}(R_b) - \text{Area}(R_a) = \mathcal{A}_f(b) - \mathcal{A}_f(a).$$

Let F be any antiderivative of f. Then F must differ from $\mathcal{A}_f$ by an additive constant C. In other words, for all x, we have

$$\mathcal{A}_f(x) - F(x) = C.$$

In particular, $\mathcal{A}_f(b) - F(b) = \mathcal{A}_f(a) - F(a) = C$, so that $\mathcal{A}_f(b) - \mathcal{A}_f(a) = F(b) - F(a)$. Thus we have shown that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The **Fundamental Theorem of Calculus** summarizes the above results:

(I) $\dfrac{d}{dx}\displaystyle\int_c^x f(t)\,dt = f(x)$. *(Area Function Theorem)*

(II) If $\displaystyle\int f(x)\,dx = F(x)$ then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

The second part of the Fundamental Theorem of Calculus (FTC-II) is the one toward which many students of mathematics develop a lasting attachment. Presented with a problem of the form "evaluate the definite integral $\int_a^b f(x)\,dx$" the student cannot seem to conceive of it as anything but a problem in antidifferentiation. Granted, if an explicit antiderivative $F(x)$ of $f(x)$ can be found, then, by FTC-II, the answer is just $F(b) - F(a)$. However, as mentioned before, for a typical function $f(x)$ drawn at random from the class of all functions, the possibility of finding a reasonably tame antiderivative $F(x)$ is remote in the extreme. In some cases, a geometric argument based on the shape of the region whose area is represented by the integral may yield an exact answer (see the quarter-circle example in Section 3.4). But in the vast majority of instances there is no option but to resort to approximate numerical methods.

In view of everything that has been said above, the optimal procedure for evaluating a definite integral $\int_a^b f(x)\,dx$ may be summarized as follows. In the first place, it is important to understand that in mathematics an exact answer is always preferable to an approximate answer. So we should explore all avenues that might yield an exact answer before resorting to approximate numerical methods:

- **(a)** Try to find an explicit antiderivative $F(x)$ of $f(x)$. Then, by FTC-II, the exact answer is $F(b) - F(a)$.
- **(b)** If (a) fails, examine the geometric shape of the region whose whose area is represented by the definite integral. Look for symmetries and special figures (triangles, circles, etc.). With any luck, it may be possible to compute the area of the region exactly.
- **(c)** If (a) and (b) fail, then one must resort to approximate numerical methods.

By the Fundamental Theorem of Calculus, the operations of integration and antidifferentiation are essentially equivalent. Through this equivalence, the basic formulas pertaining to antidifferentiation (see Section 3.3) can be converted into corresponding formulas pertaining to integration.

GENERAL INTEGRATION FORMULAS

Let f and g be functions, and let α, β, a, b and c be real numbers.

[GIF 1] If $a \leq b \leq c$ then $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ *(Split Interval Rule).*

[GIF 2] $\int_a^b (\alpha f(x) + \beta g(x))\ dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$ *(Linearity Rule).*

[GIF 3] $\int_a^b f(x)g'(x)\,dx = (f(x)g(x))\,|_a^b - \int_a^b g(x)f'(x)\,dx$ *(Integration by Parts).*

[GIF 4] $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ where $u = g(x)$ *(u-substitution Rule).*

Ex: To illustrate the u-substitution rule, consider $I = \int_1^e \frac{\ln(x)}{x}\,dx$. The substitution $u = \ln(x)$ converts the integral to $I = \int_0^1 u\,du = \left.\frac{1}{2}u^2\right|_0^1 = \frac{1}{2}$.

Ex: The u-substitution rule is useful even when an explicit antiderivative for the integrand cannot be found. For instance, consider the (improper) integral $I = \int_0^4 \frac{\sqrt{8-x}}{\sqrt{x}}\,dx$. The substitution $u = \sqrt{x}$ converts the integral to $I = 2\int_0^2 \sqrt{8-u^2}\,du$. Now, despite the fact that an antiderivative for the integrand cannot easily be found, a geometric analysis of the shape of the region, whose area is represented by the integral, reveals it to be composed of a right triangle of base and height both equal to 2 and one-eighth of a disk of radius $2\sqrt{2}$. Therefore $I = 2\left(\frac{1}{2}(2)(2) + \frac{1}{8}\pi(2\sqrt{2})^2\right) = 2\pi + 4$.

3.6 Exercises

1. Use the Newton Quotient to find a formula for $f'(x)$.
   (a) $f(x) = x^3$
   (b) $f(x) = \sqrt{x}$

2. Use the rules of derivatives to find a formula (simplified) for $f'(x)$.
   (a) $f(x) = e^x \sin^2(x)$
   (b) $f(x) = e^{\sin(x)}$
   (c) $f(x) = \ln(1 + e^x)$
   (d) $f(x) = e^{g(x)}$
   (e) $f(x) = \ln(g(x))$

3. Find the equation of the tangent line to the graph of $y = f(x)$ at the point P.
   (a) $f(x) = x\cos(x)$, $P = (0, 0)$.
   (b) $f(x) = \dfrac{2x+1}{x-1}$, $P = (2, 5)$.
4. Find all points on the graph of $y = \dfrac{x^2}{x-1}$ where the tangent line is horizontal.
5. Find an explicit simplified formula for the antiderivative.
   (a) $\displaystyle\int \frac{2x^2+1}{x}\,dx$
   (b) $\displaystyle\int x^2 e^x\,dx$
   (c) $\displaystyle\int \frac{x+17}{x^2+4x-5}\,dx$
   (d) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$
6. Evaluate the definite integral exactly by whatever method is appropriate.
   (a) $\displaystyle\int_1^3 x^2\left(4x-\frac{2}{x}\right)dx$
   (b) $\displaystyle\int_0^4 \sqrt{x^4+9x^2}\,dx$
   (c) $\displaystyle\int_{-\pi/2}^{\pi/2} (\sin(x)+\cos(x))\sin^2(x)\,dx$
   (d) $\displaystyle\int_0^5 f(x)\,dx$ where $f(x) = \begin{cases} \sqrt{8-x^2} & \text{if } 0 \le x \le 2 \\ 3x^2-10 & \text{if } x > 2 \end{cases}$
7. Evaluate the definite integral exactly, if possible, and to an accuracy of at least eight significant digits, if not possible. In each case, describe the method you used.
   (a) $\displaystyle\int_0^1 \left(x^2+1\right)^9 x\,dx$
   (b) $\displaystyle\int_0^3 2^x\,dx$
   (c) $\displaystyle\int_0^{\ln(3)} e^{\sqrt{x^2\sin^2(x)+x^2\cos^2(x)}}\,dx$
   (d) $\displaystyle\int_0^{10} e^{-\pi x^2}\,dx$
   (e) $\displaystyle\int_1^2 \sqrt{1+x^4}\,dx$
   (f) $\displaystyle\int_0^1 \frac{4}{x^2+1}\,dx$