The Five Famous Formulas of College Mathematics

6.1 The Binomial Theorem

The Binomial Theorem is the well-known formula

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\,.}$$

This formula is valid for any pair of elements x and y in any commutative ring with identity (see Section 1.5) and for any integer exponent $n \geq 1$.

The coefficients $\binom{n}{k}$, known as the *binomial coefficients*, are defined by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

It is easily verified that the binomial coefficients satisfy the recursive formula

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}.$$

Through *Pascal's Triangle*, this recursive formula can be used to build arbitrarily long lists of binomial coefficients:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & & & & \text{etc.} & & & & & & &
\end{array}$$

Observe how Pascal's Triangle is constructed. The binomial coefficient $\binom{n}{k}$ appears in Pascal's Triangle as the k-th entry ($k \geq 0$) in the n-th row ($n \geq 0$). Every entry is gotten by summing the two entries directly above it, which is precisely the rule implied by the recursive formula.

Ex: Let $x, y \in \mathbb{R}$, then, using the coefficients in the 7-th row of Pascal's Triangle, we have

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7\,.$$

Ex: Everyone's favorite differentiation formula

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad (n \ge 1)$$

can easily be proved with the aid of the Binomial Theorem.

Proof: By the Newton quotient definition of the derivative, we have

$$\begin{aligned}
\frac{d}{dx}(x^n) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} x^{n-k}h^k - x^n}{h} \\
&= \lim_{h\to 0} \frac{\sum_{k=1}^{n} \binom{n}{k} x^{n-k}h^k}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + h^2 \sum_{k=2}^{n} \binom{n}{k} x^{n-k}h^{k-2}}{h} \\
&= \lim_{h\to 0} nx^{n-1} + h\sum_{k=2}^{n} \binom{n}{k} x^{n-k}h^{k-2} \\
&= nx^{n-1}
\end{aligned}$$

6.2 The Sum of a Geometric Sequence

If $x \in \mathbb{C}$, $x \neq 1$, then

$$\boxed{\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}\,.}$$

To prove this formula, let $\sum_{k=0}^{n} x^n = S$. Then

$$\begin{aligned}
S &= 1 + x + x^2 + \cdots + x^{n-1} + x^n \\
xS &= x + x^2 + x^3 + \cdots + x^n + x^{n+1} \\
1 + xS &= 1 + x + x^2 + x^3 + \cdots + x^n + x^{n+1} \\
1 + xS &= S + x^{n+1} \\
xS - S &= x^{n+1} - 1 \\
(x-1)S &= x^{n+1} - 1 \\
S &= \frac{x^{n+1}-1}{x-1}
\end{aligned}$$

Ex: a) Let $x = 3$, $n = 7$, then $1 + 3 + 3^2 + \cdots + 3^7 = \dfrac{3^8 - 1}{3 - 1} = 3280.$

b) Let $x = -3$, $n = 7$, then $1 - 3 + 3^2 - 3^3 + \cdots - 3^7 = \dfrac{(-3)^8 - 1}{-3 - 1} = -1640.$

c) Let $x = \sqrt{2}$, $n = 7$, then $1 + \sqrt{2} + (\sqrt{2})^2 + \cdots + (\sqrt{2})^7 = \dfrac{(\sqrt{2})^8 - 1}{\sqrt{2} - 1} = 15\sqrt{2} + 15.$

d) Let $x = \frac{2}{5}$, $n = 7$, then $1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \cdots + \left(\frac{2}{5}\right)^7 = \dfrac{\left(\frac{2}{5}\right)^8 - 1}{\frac{2}{5} - 1} = \dfrac{130123}{78125}.$

e) Let $x = 1 + i$, $n = 7$, then $1 + (1+i) + (1+i)^2 + \cdots + (1+i)^7 = \dfrac{(1+i)^8 - 1}{(1+i) - 1} = -15i.$

There are a number of useful variations of the main formula. For instance, if $a, b \in \mathbb{C}$, $a \neq b$, then, setting $x = \frac{a}{b}$, we obtain

$$1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n = \frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1}.$$

Multiplying both sides by b^n and simplifying, we get

$$b^n + b^{n-1}a + b^{n-2}a^2 + \cdots + ba^{n-1} + a^n = \frac{a^{n+1} - b^{n+1}}{a - b},$$

which yields the factorization formula

$$a^{n+1} - b^{n+1} = (a - b)\left(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n\right).$$

Ex: a) $a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$.

b) $a^5 - b^5 = (a - b)\left(a^4 + a^3b + a^2b^2 + ab^3 + b^4\right)$.

c) $a^6 - b^6 = (a - b)\left(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5\right)$.

Ex: An important unsolved question in number theory is to determine whether or not there exist infinitely many primes of the form $2^q - 1$, where $q \geq 2$. Such primes are called *Mersenne primes*. For instance, $2^2 - 1 = 3$, $2^3 - 1 = 7$, and $2^5 - 1 = 31$ are Mersenne primes. It is strongly suspected that infinitely many Mersenne primes exist, but no one has yet succeeded in proving it. As an application of the above factorization formula, we will show that if $2^q - 1$ is prime, then q itself must be prime. Equivalently, it suffices to show that if q is composite then $2^q - 1$ is also composite. Suppose $q = km$ where $k \geq 2$ and $m \geq 2$. Then, by the factorization formula, with $a = 2^k$, $b = 1$, and $n = m - 1$, we have

$$\begin{aligned} 2^q - 1 &= (2^k)^m - 1 \\ &= \left(2^k - 1\right)\left((2^k)^{m-1} + (2^k)^{m-2} + \cdots + (2^k)^2 + 2^k + 1\right), \end{aligned}$$

which shows that $2^q - 1$ is composite. We conclude that every Mersenne prime must be of the form $2^p - 1$, where the exponent p is prime.

Another useful variation of the geometric sequence summation formula is gotten by letting the number of terms in the sum tend to infinity. Suppose $x \in \mathbb{C}$ and $|x| < 1$. Then, since $\lim\limits_{n\to\infty} x^{n+1} = 0$, we have

$$\begin{aligned} \sum_{k=0}^{\infty} x^k &= \lim_{n\to\infty} \sum_{k=0}^{n} x^k \\ &= \lim_{n\to\infty} \frac{x^{n+1}-1}{x-1} \\ &= \frac{-1}{x-1}. \end{aligned}$$

Thus we obtain the infinite geometric series formula:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

As noted above, this formula is valid provided $x \in \mathbb{C}$ and $|x| < 1$.

Ex: a) Let $x = \frac{2}{3}$, then $1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots = \dfrac{1}{1-\frac{2}{3}} = 3.$

b) Let $x = -\frac{2}{3}$, then $1 - \frac{2}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 + \cdots = \dfrac{1}{1+\frac{2}{3}} = \dfrac{3}{5}.$

c) Let $x = \frac{1+i}{4}$, then $1 + \left(\frac{1+i}{4}\right) + \left(\frac{1+i}{4}\right)^2 + \left(\frac{1+i}{4}\right)^3 + \cdots = \dfrac{1}{1-\left(\frac{1+i}{4}\right)} = \dfrac{6}{5} + \dfrac{2}{5}i.$

6.3 The Definiton of e

The fundamental constant e (named in honor of Euler), is defined by the formula

$$\boxed{e = \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m.}$$

With a good calculating device, the above formula can be used to estimate the value of e to any desired degree of accuracy. To an accuracy of nine decimal places $e \approx 2.718281828$. In this section it will be shown how the above formula for e arises naturally in the context of compound interest, and it will be shown how the Binomial Theorem can be used to convert the above formula into a alternate, more powerful formula for e.

Suppose a certain amount of money A is deposited in an account that pays an annual rate of interest r compounded n times per year. After a period of t years, the amount of money $A(t)$ in the account is given by the well-known compound interest formula

$$A(t) = A\left(1 + \frac{r}{n}\right)^{nt}$$

Define a new variable m as the ratio of the number of compoundings per year to the annual rate of interest. That is $m = n/r$. Then, in terms of m, the compound interest formula becomes

$$A(t) = A\left(1 + \frac{1}{m}\right)^{mrt}.$$

Now imagine that the number n of compoundings per year increases without limit. Since m is a constant multiple of n, it too increases without limit. When m is large enough, the process of crediting interest to the account takes place so frequently that, for all practical purposes, we may think of the compounding process as "continuous". The resulting formula for continuously compounded interest is

$$A(t) = \lim_{m\to\infty} A\left(1 + \frac{1}{m}\right)^{mrt},$$

which is equivalent to

$$A(t) = A\left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$

Setting

$$e = \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{m},$$

we obtain the familiar formula for continuously compounded interest, namely

$$A(t) = Ae^{rt}.$$

The above limit formula for e converges rather slowly, in the sense that, for a reasonable estimate of e to be obtained, the value of m must be taken quite large. For intance, if we take $m = 100$, the resulting estimate is accurate only to two significant digits ($e \approx 2.704813829$). If we take $m = 10,000,000$, the resulting estimate is much better, but still only accurate to seven significant digits ($e \approx 2.718281693$). Using the Binomial Theorem, we can convert the limit formula for e to a formula that converges much more rapidly.

Assume that m is a positive whole number. By the Binomial Theorem we have

$$\begin{aligned}\left(1 + \frac{1}{m}\right)^{m} &= \sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{m}\right)^{k} \\ &= \sum_{k=0}^{m}\frac{m(m-1)(m-2)\cdots(m-k+1)}{m^k}\cdot\frac{1}{k!}\end{aligned}$$

Taking the limit as $m \to \infty$, we get

$$e = \lim_{m\to\infty}\sum_{k=0}^{m}\frac{m(m-1)(m-2)\cdots(m-k+1)}{m^k}\cdot\frac{1}{k!}$$

At this point we note that

$$\lim_{m\to\infty}\frac{m(m-1)(m-2)\cdots(m-k+1)}{m^k} = 1.$$

Thus, assuming that the process of taking the limit term by term within the infinite sum can be rigorously justified, we obtain the formula

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}\,.$$

This formula converges very rapidly. For instance, if we take only the first ten terms, we obtain the approximation $e \approx 2.718281526$, which is accurate to seven significant digits.

6.4 Taylor-Maclaurin Series

The Taylor-Maclaurin series of a function $f(x)$ is given by the formula

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k\,.}$$

The validity of this formula is subject to certain conditions. The following set of conditions are sufficient but not strictly necessary (necessary conditions are difficult to establish):

(a) There exists an open interval $I = (-R, R)$ centered at $x = 0$, such that $f(x)$ possesses derivatives of all orders in I. That is, $f^{(k)}(x)$ exists for all $x \in I$ and all $k \geq 0$.

(b) For every $n \geq 0$, define $M_n = \max_{t \in I} \left| f^{(n+1)}(t) \right|$. Then $\lim_{n\to\infty} \frac{M_n R^{n+1}}{(n+1)!} = 0$.

(c) Under the above condtions, the Taylor series of $f(x)$ is valid at least for all $x \in I$.

(d) If conditions (a) and (b) hold for all $R > 0$, then the Taylor series representation of $f(x)$ is valid for all $x \in \mathbb{R}$.

Ex: Let $f(x) = \sin(x)$, then $f^{(k)}(x) = \pm\cos(x)$ or $\pm\sin(x)$, depending on the remainder of $k \pmod 4$. Thus $f(x)$ possesses derivatives of all orders in every open interval $(-R, R)$. Furthermore, since $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$ for all $x \in \mathbb{R}$, we see that $M_n = 1$ for all n. Therefore the limit in condition (b) reduces to $\lim_{n\to\infty} R^{n+1}/(n+1)! = 0$. We conclude that the Taylor-Maclaurin series representation of $f(x) = \sin(x)$ is valid for all $x \in \mathbb{R}$. To obtain the actual series, note that

$$f^{(k)}(0) = \begin{cases} 0 & \text{if } k = 4m \\ 1 & \text{if } k = 4m+1 \\ 0 & \text{if } k = 4m+2 \\ -1 & \text{if } k = 4m+3 \end{cases}$$

The above relations can best be summarized as follows: $f^{(2k)}(0) = 0$, $f^{(2k+1)}(0) = (-1)^k$, from which we obtain the Taylor-Maclaurin series representation

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Ex: Let $f(x) = \cos(x)$, then, imitating the case of $f(x) = \sin(x)$, we see that $\cos(x)$ possesses derivatives of all orders in every open interval $(-R, R)$ and that $M_n = 1$ for all n . Thus the limit in condition (b) equals 0, whereby the Taylor-Maclaurin series representation of $f(x) = \cos(x)$ is valid for all $x \in \mathbb{R}$. To obtain the actual series, note that

$$f^{(k)}(0) = \begin{cases} 1 & \text{if } k = 4m \\ 0 & \text{if } k = 4m+1 \\ -1 & \text{if } k = 4m+2 \\ 0 & \text{if } k = 4m+3 \end{cases}$$

The above relations can best be summarized as follows: $f^{(2k)}(0) = (-1)^k$, $f^{(2k+1)}(0) = 0$, from which we obtain the Taylor-Maclaurin series representation

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

Ex: Let $f(x) = e^x$, then $f^{(k)}(x) = e^x$, so that derivatives of all orders exist in any interval $(-R, R)$. Since $f^{(k)}(0) = 1$, for all k , we obtain the following Taylor-Maclaurin series representation

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

To determine the interval of validity, note that $M_n = \max_{-R<t<R} e^t = e^R$. So the limit in condition (b) becomes $\lim_{n\to\infty} e^R R^{n+1}/(n+1)! = 0$ for all $R > 0$, which inplies that the series representation is valid for all $x \in \mathbb{R}$. It is reassuring to note that, at $x = 1$, the formula gives

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

which agrees with the formula for e established in Section 6.3.

Ex: Let $f(x) = (1-x)^{-1}$, then $f^{(k)}(x) = k!(1-x)^{-(k+1)}$. All of these derivatives exist in any interval $(-R, R)$ provided that $0 < R < 1$. Since $f^{(k)}(0) = k!$, we obtain the following Taylor-Maclaurin series representation

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

Not surprisingly, this agrees with the formula for $1/(1-x)$ discussed in Section 6.2, where we showed that the formula is valid for all x in the interval $-1 < x < 1$. However, if we try to determine the formula's interval of validity by the four conditions (a)-(d) stated above, we find that we can only conclude that the formula is valid in the smaller interval $-\frac{1}{2} < x < \frac{1}{2}$. Is this a contradiction? No, the four conditions stated above are sufficient, but not strictly necessary for validity. In other words, the interval of validity implied by those four conditions is guaranteed only to be a sub-interval of the full interval of validity. The full interval of validity may be larger.

6.5 Euler's Polar Angle Formula

Let θ be any real number, then

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)\,.}$$

The applications of this formula are numerous and far-reaching. We will explore a few of these applications, mainly by way of examples. To prove the formula, consider the Taylor-Maclaurin series of the natural exponential function:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

It can be shown that this series converges not just for all real values of x, but for all $z \in \mathbb{C}$. Thus the definition of the natural exponential function can be extended to the entire complex plane by way of the formula

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

In particular, if $t \in \mathbb{R}$, then

$$\begin{aligned} e^{it} &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{i^k t^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{i^{2k} t^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1} t^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \\ &= \cos(t) + i\sin(t). \end{aligned}$$

In the context of trigonometry it is customary to use the polar angle θ in place of the generic parameter t, so that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. In particular, according to this formula, every point $(\cos(\theta), \sin(\theta))$, on the unit circle, can be identified with the exponential expression $e^{i\theta}$.

Ex: **(Trigonometric Identities)** Since $e^{i(a+b)} = e^{ia}e^{ib}$, we have

$$\begin{aligned} \cos(a+b) + i\sin(a+b) &= (\cos(a) + i\sin(a))\,(\cos(b) + i\sin(b)) \\ &= (\cos(a)\cos(b) - \sin(a)\sin(b)) + i\,(\cos(a)\sin(b) + \sin(a)\cos(b))\,. \end{aligned}$$

By equating real and imaginary parts, we obtain the familiar trigonometric identities:

$$\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \sin(a+b) &= \cos(a)\sin(b) + \sin(a)\cos(b). \end{aligned}$$

Ex: **(De Moivre's Theorem)** Decades before Euler was born, a close friend of Isaac Newton by the name of Abraham De Moivre (1667-1754) discovered the famous identity

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

However, in light of Euler's polar angle formula, De Moivre's identity loses all of its mystery and charm, for it asserts simply that $(e^{i\theta})^n = e^{i(n\theta)}$.

Ex: **(Euler's Identity)** The famous identity

$$e^{i\pi} + 1 = 0$$

is easily derived from Euler's polar angle formula, by setting $\theta = \pi$. This identity has been known to invoke flights of mystical speculation, for it weaves together, in a single relation, the five fundamental constants of mathematics (e, i, π, 1, 0) and the three fundamental operations of arithmetic (addition, multiplication and exponentiation).

Ex: **(Roots of unity)** Let $n \geq 1$. By the Fundamental Theorem of Algebra, the polynomial $f(z) = z^n - 1$ factors completely as a product of n linear factors:

$$z^n - 1 = (z - \omega_0)(z - \omega_1)\cdots(z - \omega_{n-1}).$$

The roots $\omega_0, \omega_1, \ldots, \omega_{n-1}$ are called the n-th roots of unity. By the geometry of the unit circle, we have $e^{i\phi} = 1$ if and only if ϕ is an integer multiple of 2π. Thus, $e^{in\theta} = 1$ if and only if $\theta = 2\pi k/n$, where $k \in \mathbb{Z}$. Evidently, for $0 \leq k \leq n-1$, the points on the unit circle given by

$$\begin{aligned} \omega_k &= e^{2\pi ik/n} \\ &= \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right). \end{aligned}$$

are distinct and must therefore comprise the complete set of n-th roots of unity. For instance, the three cube roots of unity are given by $\omega_k = e^{2\pi ik/3}$, $k = 0, 1, 2$. That is:

$$\omega_0 = 1, \quad \omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \omega_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Ex: **(A Paradox)** This example illustrates the perils of applying formulas and operations without paying careful enough attention to their domain of definition. By calculator, one can verify that $e^{2\pi} \approx 535.4916555$. However, we will "prove " that $e^{2\pi} = 1$. To begin, since e^{π} is a positive real number, we can find positive real numbers a and b such $e^{\pi} = a/b$. Raising both sides to the power i, we get $a^i/b^i = e^{i\pi} = -1$. Thus $a^i = -b^i$. Squaring both sides gives $a^{2i} = b^{2i}$. Raising both sides to the power i yields $a^{-2} = b^{-2}$. Therefore $a^2 = b^2$, so that $(a/b)^2 = 1$. But since $a/b = e^{\pi}$, we conclude that $e^{2\pi} = 1$. What's wrong with this "proof"?

6.6 Exercises

1. Evaluate each of the following exactly
   (a) $\binom{10}{4}$
   (b) $\binom{10}{6}$
   (c) $\binom{20}{6} - \binom{19}{5}$
   (d) $\binom{50}{18} \div \binom{48}{16}$
   (e) $\displaystyle\sum_{k=0}^{10} \binom{10}{k} 2^k$

2. Use the Binomial theorem to expand and simplify
   (a) $(x+1)^8$
   (b) $(x+1)^8 - (x-1)^8$

3. Prove the following formulas
   (a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$
   (b) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

4. Use the identity $(1+i)^{8n} = 16^n$ to prove that $\displaystyle\sum_{k=0}^{4n-1} (-1)^k \binom{8n}{2k+1} = 0.$

5. Evaluate each of the following sums exactly
   (a) $\displaystyle\sum_{k=0}^{8} (-1)^k \frac{2^k}{5^k}$
   (b) $\displaystyle\sum_{k=0}^{\infty} 2^{-k/2}$
   (c) $\displaystyle\sum_{k=3}^{\infty} \frac{2^{2k+1}}{5^{k-1}}$
   (d) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k - 2^k}{6^k}$
   (e) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{2i}\right)^{2k+1}$

6. Find a complete factorization for
   (a) $x^9 - y^9$
   (b) $x^{11} - 1$

7. The first three Mersenne primes are 3, 7, 31. Find the next three Mersenne primes.

8. Suppose \$5,000 is deposited in an account that pays an annual rate of interest of 9.5% compounded continuously. How many years (rounded to 4 significant digits) would the depositor have to wait for the account to grow to \$25,000?

9. What is the minimum number of terms of the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ required to estimate e to an accuracy of 10 significant digits. Display your calculation in tabular form.

10. Find a formula (appropriately simplified) for the Taylor-Maclaurin series of the given function:
    (a) $f(x) = \ln(1-x)$
    (b) $f(x) = \frac{1}{2}(e^x + e^{-x})$
    (c) $f(x) = \sin(2x)$
    (d) $f(x) = \dfrac{1}{1-x^2}$
    (e) $f(x) = e^{-x^2}$

11. Simplify and express each of the following in the form $a + bi$:
    (a) $e^{i\pi/2}$
    (b) $e^{i\pi/4}$
    (c) $\dfrac{1}{2}\,(e^{i\pi/6} + e^{-i\pi/6})$
    (d) $\dfrac{1}{2i}\,(e^{i\pi/6} - e^{-i\pi/6})$

12. Find all of the 12-th roots of unity and express each of them in the form $a + bi$. Using a ruler and compass, sketch a diagram to show how the 12-th roots of unity are distributed on the unit circle.

13. Given that Euler's formula $e^{it} = \cos(t) + i\sin(t)$ is valid for all $t \in \mathbb{R}$, evaluate 2^i.

14. Prove the formulas:
    (a) $\cos(x) = \dfrac{1}{2}\,(e^{ix} + e^{-ix})$
    (b) $\sin(x) = \dfrac{1}{2i}\,(e^{ix} - e^{-ix})$

15. For $n \geq 2$, evaluate the sum $\displaystyle\sum_{k=0}^{n-1} e^{2\pi ik/n}$.